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THE SHELL WITH DOUBLE CURVATURE CONSIDERED AS A PLATE ON AN ELASTIC FOUNDATION

U.D.C. 624.074.4

From the differential equation of the doubly curved shell in terms of the displacement components u, v and w it is apparent that, in so far as the flexural phenomena are concerned, the shell can be considered as a flat plate on an elastic foundation, at any rate in a region – of sufficiently small size to be regarded as quasi-Euclidean – around the vertex of the osculating paraboloid. The modulus of the elastic foundation is dependent on the shell thickness and the principal curvatures of the middle surface. The actual load on the plate must, however, be increased by a deformation load depending on the displacements u and v in the middle surface.
Various aspects of this approach to the problem are verified by reference to simple examples and to the results of tests on a large model of an equilateral hyperbolic paraboloid shell.

0 Introduction

It is a well-known fact that a curved shell can resist loading acting perpendicularly to the middle surface by the agency of membrane forces. In those cases where the membrane reactions can be resisted and the corresponding deformations of the shell can freely take place, a force distribution pattern constituted solely by membrane forces is a good approximation of the transmission of forces that actually occurs – for a statically possible stress distribution in which bending and torsion are avoided will come close to producing the minimum strain energy in the structure. The usual procedure, therefore, is to begin by calculating the distribution of forces in the shell according to the membrane theory. Then a set of correcting forces will have to be applied so as to take the best possible account of the boundaries (edge beams or supports) of the shell, where the deformations due to the said state of membrane stress are prevented from freely developing. Let us suppose the edge member to be detached from the shell, thus enabling these deformations to develop freely: the edge of the shell and the edge member would then no longer fit at their junction. A proper fit can be obtained only if the edge members exerts forces (normal and shear forces) and moments (bending and torsion moments) on the edge of the shell and if, conversely, the shell exerts opposite forces and moments on the edge member in such a manner that the additional deformations associated with these forces and moments enable complete adaptation (structural fit) to be achieved. The calculation of these edge disturbances, which must be superimposed upon the state of membrane stress, is done with the aid of the so-called flexural theory of shells.

Simplified flexural theories are already available for shells shaped according to simple mathematical surfaces such as the spherical and the cylindrical shell. A. L. BOUMA has applied a theory of this kind to elliptic and hyperbolic paraboloid shells bounded by four principal parabolas.[1])

Even these simplified theories undeniably involve a fairly considerable amount of arithmetical computation; and for doubly curved shells of arbitrary shape or combinations thereof (which may or may not be bounded by elastically deflecting and/or elastically rotating edge beams), such as are rather frequently employed in modern architecture, there is as yet no serviceable theory for the analysis of flexural disturbances.

The distribution of the edge disturbance moment from the edges towards the interior of the shell surface is known in general to have the character of a damped wave. An obvious question is whether it would not fundamentally be possible to calculate the approximate magnitude of this edge disturbance moment by means of the theory of the elastically supported beam – for in a beam of that kind the edge disturbances display a damped-wave distribution. Thus J. W. GECKELER [2]), as long ago as 1926, gave an approximate analysis for the edge disturbance in a spherical shell which leads to the theory of the elastically supported beam. Furthermore, in 1953 K. HRUBAN [3]) analysed the edge disturbances along the curved edges of a number of hyperbolic paraboloid north-light shells with the theory of the elastically supported beam, while in 1958 W. S. WLASSOW [4]), with reference to the spherical shell and the equilateral hyperbolic paraboloid shell, called attention to the analogy with the elastically supported plate.

Finally, for the estimation of certain edge disturbance moments associated with the structural research on the Philips Pavilion at the Brussels World Exhibition in 1958, the present author also proposed a method of analysis based on the theory of the elastically supported plate.[5])

If permissible, the method of analysis based on the elastically supported plate or beam undoubtedly constitutes a considerable simplification. This is because the differential equation of eighth order on which the flexural theory of shells is based is replaced by an equation of fourth order, which provides a simpler means of gaining an insight into the anticipated edge disturbances, this being of great importance to the designer.

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[1]) BOUMA, A. L. Some applications of the bending theory regarding doubly curved shells. Proc. Symposium on Theory of Thin Elastic Shells (I.U.T.A.M.), North-Holland Publ. Co., Amsterdam, 1960.

[2]) GECKELER, J. W. Über die Festigkeit achsensymmetrischer Schalen. Forschungsarbeiten Ing. wesen, No. 276, Berlin, 1926.

[3]) HRUBAN, K. The general theory of saddle-shaped shells (Czech), Techn. Univ. Brno, 1953.

[4]) WLASSOW, W. S. Allgemeine Schalentheorie und ihre Anwendung in der Technik, Akademie-Verlag, Berlin, 1958, pp. 330 and 372.

[5]) VREEDENBURGH, C. G. J. The hyperbolic-paraboloidal shell and its mechanical properties, Philips Technical Review, Vol. 20, No. 1, 1958/1959.

On the other hand, when applying an approximate method of analysis it is essential to have a clear conception of the quantities that are being neglected and of the assumptions made, in order to be able to judge whether useful results are likely to be obtained in any particular case.

1 The doubly curved shell and the elastically supported plate

Consider a point O in the middle surface of a doubly curved shell of arbitrary shape. This point is the origin of a rectangular co-ordinate system XYZ. Let the plane OXY coincide with the plane tangential to the middle surface at O and let the axis OZ be directed along the normal (Fig. 1).

In the immediate vicinity of O the shape of the middle surface can, as we know, be replaced by a paraboloid, the so-called osculating paraboloid, which has the following equation:

$$2z = \frac{x^2}{r_x} + \frac{2xy}{r_{xy}} + \frac{y^2}{r_y} \quad \ldots \ldots \ldots \ldots \quad (1)$$

or

$$2z = k_x x^2 + 2k_{xy}xy + k_y y^2 \quad \ldots \ldots \ldots \ldots \quad (2)$$

In these equations r_x, r_y and r_{xy} respectively denote the radii of curvature of the middle surface at O in the x and y directions and the radius of torsion for these directions, while k_x, k_y and k_{xy} respectively denote the curvatures in the x and y directions and the warp for these directions.

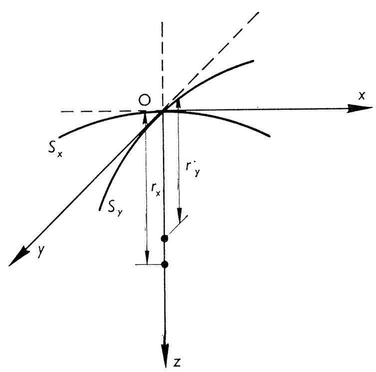

Fig. 1. OZ = normal at point O of the middle surface; S_x and S_y = normal sections; r_x and r_y = radii of curvature of S_x and S_y at O (positive if the centre of curvature has a positive co-ordinate z); u, v and w = displacement components (positive if the displacements take place in the positive directions of the co-ordinate axes).

From (2) we obtain by differentiation:

$$\left.\begin{aligned} k_x &= \frac{\partial^2 z}{\partial x^2} \\ k_{xy} &= \frac{\partial^2 z}{\partial x \partial y} \\ k_y &= \frac{\partial^2 z}{\partial y^2} \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (3)$$

The equation of equilibrium for a shell element at O in the z-direction is:

$$n_x \frac{\partial^2 z}{\partial x^2} + 2n_{xy}\frac{\partial^2 z}{\partial x \partial y} + n_y \frac{\partial^2 z}{\partial y^2} + p_z = K\Delta\Delta w \quad \ldots \ldots \quad (4)$$

For $\nu = 0$ we furthermore have:

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} - \frac{w}{r_x} = \frac{n_x}{E\delta} \\
\varepsilon_y &= \frac{\partial v}{\partial y} - \frac{w}{r_y} = \frac{n_y}{E\delta} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2w\,\frac{\partial^2 z}{\partial x \partial y} = \frac{n_{xy}}{G\delta} = \frac{2n_{xy}}{E\delta}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (5)$$

or:

$$\left.\begin{aligned}
n_x &= E\delta \left[\frac{\partial u}{\partial x} - w\,\frac{\partial^2 z}{\partial x^2}\right] \\
n_y &= E\delta \left[\frac{\partial v}{\partial y} - w\,\frac{\partial^2 z}{\partial y^2}\right] \\
n_{xy} &= E\delta \left[\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - w\,\frac{\partial^2 z}{\partial x \partial y}\right]
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (6)$$

Notation

x, y and z	co-ordinates of a point in the middle surface
u, v and w	displacement components of a point in the middle surface in the directions x, y and z respectively
r_x, r_y and r_{xy}	radii of curvature of the middle surface in the directions x and y respectively, and the radius of torsion for these directions
$\frac{1}{r_x} = k_x = \frac{\partial^2 z}{\partial x^2}$, $\frac{1}{r_y} = k_y = \frac{\partial^2 z}{\partial y^2}$ and $\frac{1}{r_{xy}} = k_{xy} = \frac{\partial^2 z}{\partial x \partial y}$:	curvatures of the middle surface in the directions x and y respectively, and the warp for these directions
r_1, r_2	principal radii of curvature of the middle surface
$k_1 = \frac{1}{r_1}$, $k_2 = \frac{1}{r_2}$	principal curvatures of the middle surface
n_x, n_y and n_{xy}	shell normal forces in the directions x and y respectively, and the shell shear force
m	shell moment
q	shell shear force (transverse shear)
ε_x, ε_y and γ_{xy}	strains in the directions x and y respectively, and angular deformation due to shear (deviation from right angle)
E, G	elastic modulus and shear modulus
ν	Poisson's ratio
δ	shell thickness
$K = \frac{E\delta^3}{12(1-\nu^2)}$	flexural rigidity of the shell
c	modulus of the elastic foundation
λ	characteristic length
p_z	load per unit area of the middle surface in the direction of the normal
p_z	deformation load
Δ	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

By substituting the relations (6) into (4) and having regard to (3), we obtain:

$$K\Delta\Delta w = p_z + E\delta\left[k_x\frac{\partial u}{\partial x} + k_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + k_y\frac{\partial v}{\partial y}\right] - E\delta(k_x^2 + 2k_{xy}^2 + k_y^2)w \quad (7)$$

Essentially this equation is valid for the shell in the immediate vicinity of O, but it is approximately valid also for points of the shell situated not too far from O. For practical purposes we may consider this region to be bounded by a circle having O as its centre and having a radius equal to half the smallest principal radius of curvature at O. Then the ratio between the rise and the chord of all the normal sections of the osculating paraboloid will be smaller than $^1/_7$ so that the corresponding region of the shell can, as a rule, be assumed to be quasi-Euclidean.

Putting:

$$E\delta\left[k_x\frac{\partial u}{\partial x} + k_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + k_y\frac{\partial v}{\partial y}\right] = \bar{p}_z \quad \ldots \quad (8)$$

$$E\delta(k_x^2 + 2k_{xy}^2 + k_y^2) = c \quad \ldots \quad (9)$$

we can now write equation (7) as follows:

$$K\Delta\Delta w = p_z + \bar{p}_z - cw \quad \ldots \quad (10)$$

The latter equation is seen to be the differential equation of a flat plate, having a stiffness K, supported on an elastic foundation with a modulus of reaction (or modulus of foundation) c, this plate being subjected to a real load p_z and an additional load $\bar{p}_z$ which is dependent on the differential quotients of the displacement components u and v (the displacements in the middle surface of the shell).

This last-mentioned load will be called the "deformation load". The values of $\bar{p}_z$ and c are invariant, i.e., they are independent of rotation of the co-ordinate system XYZ about the origin O.

Choosing the axes OX and OY so as to coincide with the principal directions of curvature of the middle surface at O, we obtain with $k_x = k_1$, $k_y = k_2$ and $k_{xy} = 0$:

$$\bar{p}_z = E\delta\left[k_1\frac{\partial u}{\partial x} + k_2\frac{\partial v}{\partial y}\right] \quad \ldots \quad (11)$$

$$c = E\delta\,[k_1^2 + k_2^2] \quad \ldots \quad (12)$$

If the factor ν is also taken into account, the relations (11) and (12) become:

$$\bar{p}_z = \frac{E\delta}{1-\nu^2}\left[(k_1+\nu k_2)\frac{\partial u}{\partial x} + (k_2+\nu k_1)\frac{\partial v}{\partial y}\right] \quad \ldots \quad (13)$$

$$c = \frac{E\delta}{1-\nu^2}\,[k_1^2 + 2\nu k_1 k_2 + k_2^2] \quad \ldots \quad (14)$$

while the plate stiffness in (10) becomes:

$$K = \frac{1}{12}\frac{E\delta^3}{1-\nu^2} \quad . \qquad (15)$$

Summarising, it is apparent from the foregoing that the displacements w in the direction of a normal at a point of a quasi-Euclidean portion of an arbitrary shell are the same as those of a flat plate on an elastic foundation, the edge support conditions of the plate being the same as those of the shell. The modulus of the foundation is determined by the local principal radii of curvature and the local thickness of the plate according to formula (12) or (14). Besides the normal load p_z it is, however, necessary also to apply an additional load $\bar{p}_z$ (the so-called deformation load) in the direction of the normal, according to formula (11) or (13).

In contrast with the load p_z, which is known, as given quantity, the deformation load is, generally speaking, unknown. Often, however, if the latter is not neglected, it can be approximately determined; some examples of this will be given in due course. In the design of a shell it is, of course, advisable to endeavour to keep the displacements u and v as small as possible by employing sufficiently rigid edge members of the shell – so that, in particular, no extensionless deformations [1]) will occur – or by the use of appropriately chosen prestresses.

In some cases these displacements are even zero, e.g., in a spherical shell or a cylindrical shell subjected to a constant normal load and supported in the manner indicated in Fig. 2, so that no edge disturbances occur. Another example of a case where the deformation load is zero is afforded by the cone of revolution subjected

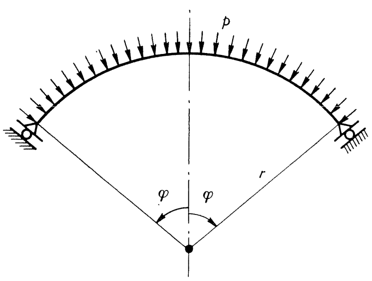

Fig. 2. Spherical shell or circular cylindrical shell subjected to uniformly distributed radial load, without edge disturbances.

to axially symmetrical loading when $\nu = 0$. If the X-axis is made to coincide with a generating line of the surface, then obviously $k_1 = 0$ and $\frac{\partial v}{\partial y} = 0$, so that $\bar{p}_z = 0$ according to (13).

The application of the plate analogy becomes particularly attractive in a case where the edge disturbance moments are confined to four strips along the

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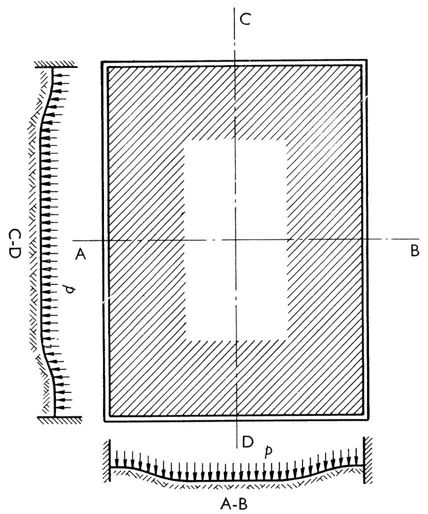
[1]) For the significance of "extensionless deformations" see: Bouma, A. L., Stijfheid en sterkte van schalen, Waltman, Delft, 1960.

Fig. 3. Hyperbolic paraboloid shell, bounded by four generating lines and subjected to uniformly distributed load, considered as an elastically supported plate. The edge members are very stiff beams. The shaded area is the edge disturbance zone. In the region surrounded by this zone the deflection w of the plate is constant and the bending moment is therefore zero in all directions. The edge disturbance zone has a maximum width of 4λ.



edges, as indicated in Fig. 3 for a rectangular hyperbolic paraboloid shell bounded by generating lines.

Consider a "beam" cut from the plate perpendicularly to the edge and passing through the zone free from disturbances. Equation (10) can then be simplified to:

$$K\frac{d^4w}{dx^4} = p_z + \bar{p}_z - cw \quad \dots \quad (16)$$

if the X-axis is chosen in the direction of the axis of the beam.

Having regard to (12), we may put:

$$\frac{c}{K} = \frac{4}{\lambda^4} = \frac{E\delta(k_1^2+k_2^2)}{1/12E\delta^3}$$

Hence:

$$\lambda^4 = \frac{\delta^2}{3(k_1^2+k_2^2)} \quad \dots \quad (17)$$

$$\lambda = \frac{0{,}76\sqrt{\delta}}{\sqrt[4]{k_1^2+k_2^2}} \quad \dots \quad (18)$$

The quantity λ, which is known from the theory of the beam on an elastic foundation and has the dimension of a length, is sometimes termed the characteristic length. It can often be assumed that, depending on the mode of support, the edge disturbances will, for practical purposes, no longer be perceptible beyond a strip having a width ranging from λ to 4λ along the edges of the shell.

2 The foundation modulus c

Consider a circular cylindrical shell rectangular on plan, which is subjected to a uniformly distributed radial load p acting over its entire surface and which is so supported at the edges as to be free from disturbances.

Let the radius of the circle be r (Fig. 2). Since, as already stated, in this case the displacements u and v are both everywhere zero, the deformation load is zero. Furthermore, the displacement w is constant, and – having regard to (10) – we therefore find:

$$w = \frac{p}{c} \qquad (19)$$

Introducing $k_1 = \frac{1}{r}$ and $k_2 = 0$ we then obtain, according to (14):

$$c = \frac{E\delta}{(1-\nu^2)r^2}$$

so that finally:

$$w = \frac{(1-\nu^2)pr^2}{E\delta} \qquad (20)$$

In this case the displacement can also be calculated directly, as follows. The compressive (tangential) "hoop" stress in the shell is:

$$\sigma = \frac{pr}{\delta} \qquad (21)$$

As the lateral contraction is, in the present case, prevented in the longitudinal direction of the shell, the compressive stress in the direction of the generating line will be:

$$\sigma' = \nu\sigma \qquad (22)$$

The specific compressive strain along the circle is therefore:

$$\varepsilon = \frac{(1-\nu^2)\sigma}{E} = \frac{(1-\nu^2)pr}{E\delta} \qquad (23)$$

Since this strain is equal to $\frac{w}{r}$ (inasmuch as $u = v = 0$), we arrive at the result embodied in (20).

Now let us consider a spherical shell, with a circular base, subjected to a uniformly distributed radial load p acting upon its entire surface. Let r be the radius of the sphere and let the shell be again so supported at the edge as to be free from disturbances.

In this case too all the points on the sphere will undergo only (constant) radial displacements w, so that u and v, and therefore also the deformation load, will be zero at all points.

The displacement w is obviously again expressed by the formula (19), but the foundation modulus c for the sphere is different from that for the cylindrical shell.

Introducing $k_1 = k_2 = \frac{1}{r}$ we now have, according to (14):

$$c = \frac{E\delta}{1-\nu^2}\left[\frac{1}{r^2} + \frac{2\nu}{r^2} + \frac{1}{r^2}\right] = \frac{2E\delta}{(1-\nu)\,r^2}$$

so that:

$$w = \frac{(1-\nu)pr^2}{2E\delta} \qquad \qquad (24)$$

By direct calculation we obtain:

$$\sigma = \frac{pr}{2\delta} \text{ and } \varepsilon = \frac{(1-\nu)pr}{2E\delta} = \frac{w}{r},$$

i.e., w is found to have the same value as that given by (24).

These simple calculations may be regarded as providing a check on the correctness of the formula (14) for the foundation modulus c if the shell is considered as a plate on an elastic foundation and the deformation load is zero. Since – as will be apparent in due course – both the actual load and the foundation modulus of the analogous plate are modified as a result of taking the deformation load into account, the foundation modulus according to (12) or (14) (which therefore relates to the case where the deformation load is zero) will be called the "primary" foundation modulus. The modulus modified by the deformation load will be called the "secondary" foundation modulus.

3 The deformation load

Consider a circular cylindrical shell having an elongated rectangular shape on plan and subjected to a uniformly distributed radial load p acting upon its entire surface. In this case, however, the shell is assumed to be completely restrained at all its edges. Let the Y-axis be chosen in the longitudinal direction of the shell, and the X-axis in the transverse direction (Fig. 4).

Evidently, at a sufficiently great distance from the transverse edge AB the disturbance emanating from this edge and proceeding in the Y-direction will have been practically damped out, so that we can write here $\frac{\partial w}{\partial y} = 0$. Now consider a transverse strip of unit width cut from the shell in this region. For the deformation load we then obtain, on introducing $k_1 = \frac{1}{r}$ and $k_2 = 0$ and having regard to (11):

$$\bar{p}_z = \frac{E\delta}{r}\cdot\frac{\partial u}{\partial x} \qquad \qquad (25)$$

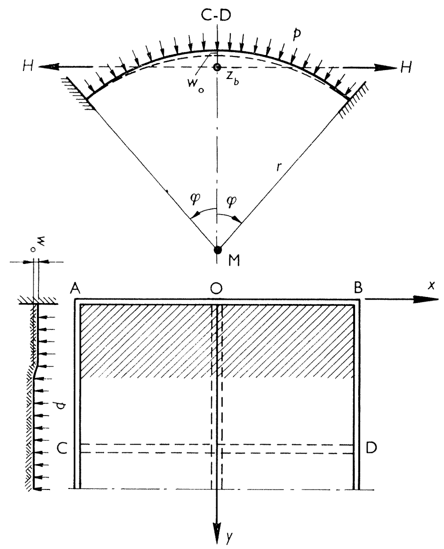


Fig. 4. Long circular cylindrical shell rectangular on plan, completely restrained at all the edges and subjected to uniformly distributed radial load. Outside the edge disturbance zone of the short edge (shaded area) a transverse strip cut out of the shell no longer behaves as a beam on an elastic foundation, but as an arch fixed at both ends, in which the line of action of the horizontal tensile force passes through the elastic centre. This is caused by the deformation load.

and having regard to the first equation (5):

$$\bar{p}_z = \frac{E\delta}{r}\left[\frac{n_x}{E\delta} + \frac{w}{r}\right] \quad \ldots\ldots\ldots \quad (26)$$

As the effect of bending on n_x is very slight in the present case, we can write $n_x = -pr$, so that equation (10) becomes:

$$K\Delta\Delta w = p - p + \frac{E\delta w}{r^2} - cw$$

which, on introducing

$$c = \frac{E\delta}{r^2} \text{ and } \frac{\partial w}{\partial y} = 0$$

gives:

$$\frac{\partial^4 w}{\partial x^4} = 0 \quad \ldots\ldots\ldots \quad (27)$$

In this case the loading of the analogous plate and the foundation modulus have both become zero in consequence of the deformation load, and the displacements w of the cut-out strip CD are governed only by an edge load at the points of restraint C and D of an unloaded beam. This restraint reaction is not difficult to determine. It is a horizontal tensile force H passing through the centre Z_b of the arch, Z_b being located at a distance $\frac{r \sin \varphi}{\varphi}$ from the centre M.

For $\nu = 0$ the magnitude of H can be calculated from the relation:

$$H \frac{12 r^3}{E\delta^3}\left(\varphi + {}^1/_2 \sin 2\varphi - \frac{2 \sin^2\varphi}{\varphi}\right) = \frac{2pr^2}{E\delta} \sin \varphi \quad \ldots\ldots \quad (28)$$

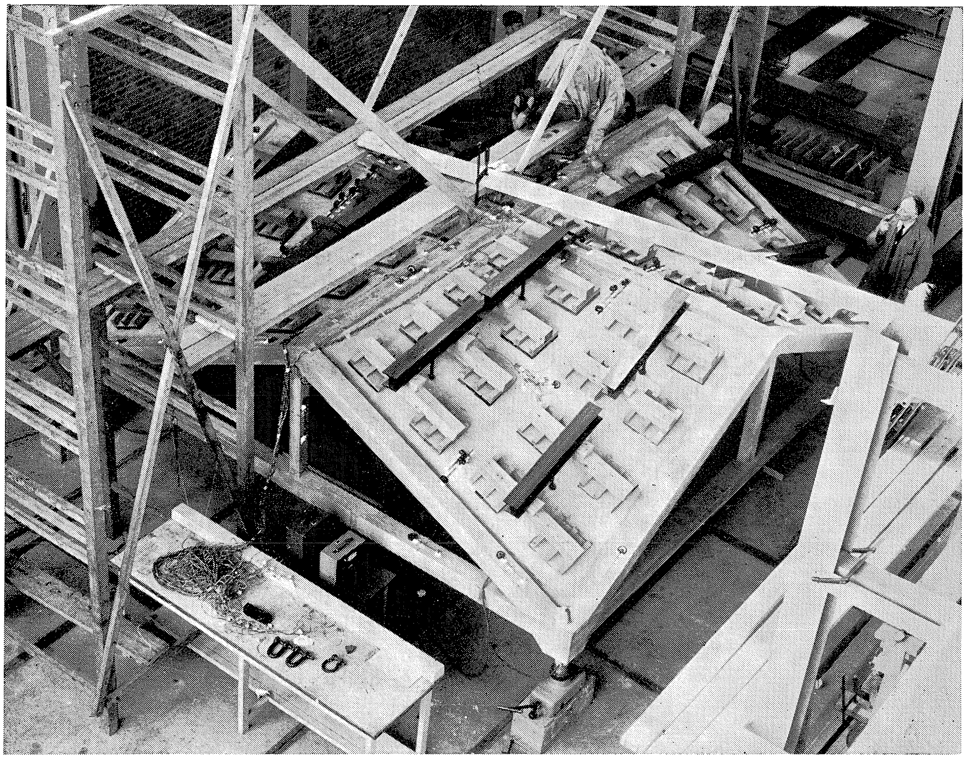


Fig. 5. General view of test arrangement for a hyperbolic paraboloid shell in the Stevin Laboratory. Overall dimensions of the model on plan 5 m × 5 m.

If H is known, the displacement w_0 at the crown of the arch can be calculated. By considering an elastically supported beam strip with its axis along OY we then obtain – with $\lambda = 0.76\sqrt{r\delta}$ according to (18) – the following expressions for the moment and for the shear in the shell at O [1]:

$$m_0 = -\frac{E\delta^3 w_0}{6\lambda^2} \qquad (29)$$

$$q_0 = \frac{E\delta^3 w_0}{3\lambda^3} \qquad (30)$$

Mention should be made of an interesting case where it was possible to check the effect of the deformation load against the actual behaviour, namely, the research conducted on a large model of a hyperbolic paraboloid shell in the Stevin Laboratory, Delft, by C. VAN DER SCHENK in 1958.[2]) The structure in question was a system of four equilateral hyperbolic paraboloid shells, each of

[1]) See also: Randstoringen bij axiaalsymmetrisch belaste omwentelingsschalen, IBC-Mededelingen, Vol. 6, No. 1, January 1958.

[2]) VAN DER SCHENK, C. Onderzoek naar de spanningsverdeling en de sterkte van een hypparschaal, Part II, Techn. Univ., Delft, 1958.

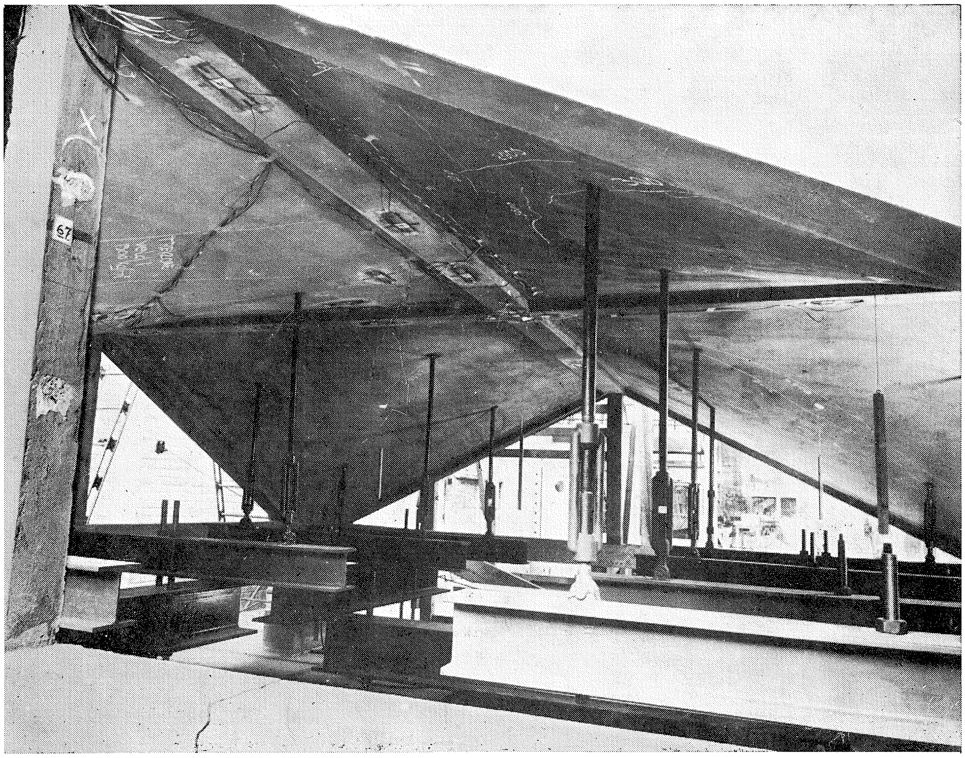


Fig. 6. Underside view of hyperbolic paraboloid shell with loading devices. Shell thickness 3 cm. Cross-sectional dimensions of hangers 10 cm × 10 cm, other edge members 15 cm × 15 cm.

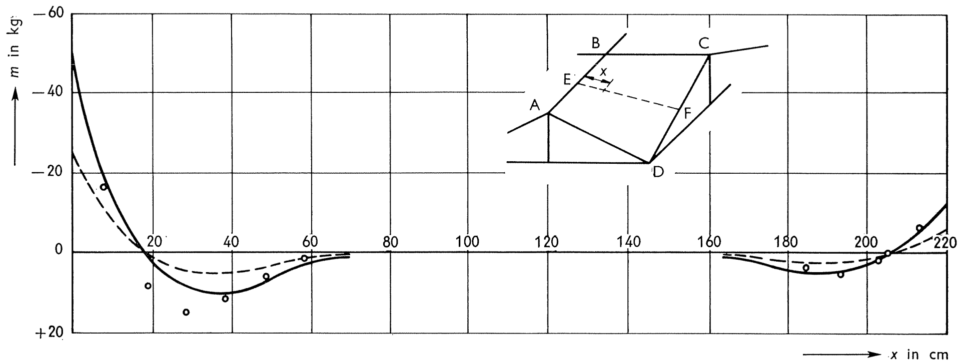


Fig. 7. Reinforced concrete hyperbolic paraboloid shell with all four bays carrying a uniformly distributed load $p = 1000$ kg/m²; $\lambda = 22.4$ cm at B.

EF considered as a beam on an elastic foundation, completely restrained at E and elastically restrained at F.

o measured shell moments

— — — calculated without deformation load

——— calculated with deformation load equal to p

which was square on plan and which were interconnected by stiff ridge beams, while the inclined edges were also provided with stiff beams (see Figs. 5, 6 and 7).

Consider the case where the four bays carry a uniformly distributed load $p = 1000$ kg/m^2 over the entire surface.

Having regard to the characteristic length (22.4 cm at the vertex), it was anticipated that the case envisaged in Fig. 3 would occur. Hence the elastically supported plate was locally replaced by an elastically supported beam. Consider a beam strip of unit width cut out of the bay ABCD along the generating line EF (Fig. 7). This beam can be regarded as completely restrained at the ridge beam. On the other hand, it is elastically restrained at its junction with the edge beam at F. Let x be the distance measured to the completely restrained end of the strip. The distribution pattern of the moment in the elastically supported beam for a uniformly distributed load p will then, as we know, be given by:

$$m_x = -{}^1/_2 p\lambda^2 \, e^{-\frac{x}{\lambda}} \sqrt{2} \cos\left(\frac{x}{\lambda} + \frac{\pi}{4}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

Neglecting the deformation load, we obtain with $p = 0.1$ kg/cm^2 and $\lambda = 22.4$ cm the following value for the restraint moment at E : $m_i = -25.1$ kg. The maximum positive moment occurs at $x = \frac{\pi}{2}\lambda$ and is 5.2 kg.

In Fig. 7 the distribution curve of the moments calculated in this way emanating from E is shown dashed.

The moments emanating from the elastically restrained support F can similarly be calculated and are also shown dashed in Fig. 7. On comparing the values calculated by neglecting the deformation load with the measured actual moments (indicated by the small circles in Fig. 7) we see that there is indeed good agreement in the pattern of the distribution, but that the calculated values are much too low. If, to allow for the effect of the deformation load, we consider twice as large an actual load on the shell (load factor = 2), we obtain the curve drawn as a full line in Fig. 7; this curve is in better agreement with the measured values. As emerged from a further investigation of the problem, the order of magnitude of the deformation load could, in the present case, be explained with the aid of the measured strains of the edge members, from which it was possible to calculate average values for $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$.

As regards the peak values of the moments at E and F it is to be noted that these are usually not so serious as they may appear to be.

For one thing, acute angles are always more or less chamfered in actual practice. Furthermore, if the limit of proportionality of the material is exceeded, a stress peak will be levelled down, while a residual stress of opposite sign will be produced on unloading. Also, if the load on the shell under present consid-

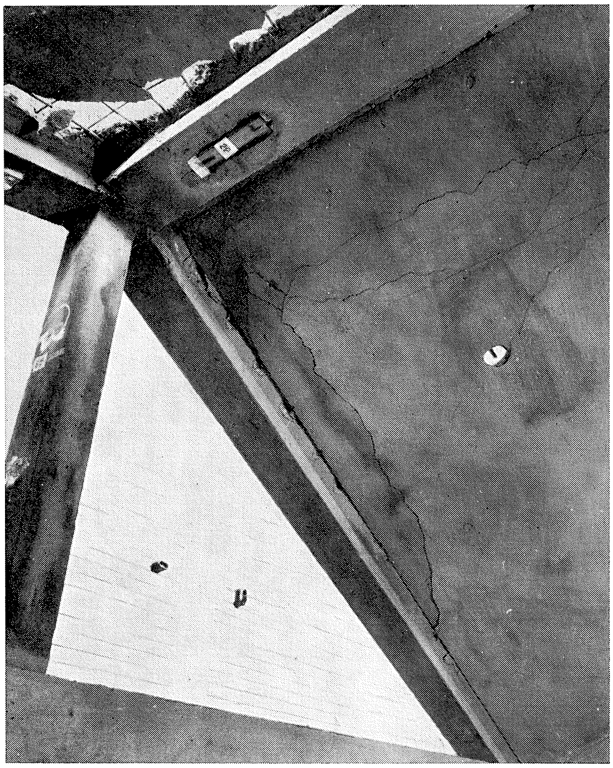


Fig. 8. Underside view of a fractured bay, showing cracks extending approximately parallel to its edges.

eration is gradually increased, the first cracks are likely to occur along the edge members. With further increased loading these cracks will function as linear hinges. The elastically supported beam EF (Fig. 7) will than have hinged bearings at both ends, with the result that the maximum positive moments will, as we know, occur, at distances $\frac{\pi}{4}\lambda$ from the ends. In the ultimate condition (i.e., at failure) a second set of yield- lines, parallel to the edges, will therefore have to develop at these points. In the test this was actually found to be so (see Fig. 8).

We can utilise this knowledge for calculating by means of the yield-line theory the ultimate load of the shell loaded over its entire surface.

For the sake of completeness it should, finally, be mentioned that the total failure of the shell in question did not occur until a load of over 6000 kg/m^2 was reached!

4 Consideration of a diametral beam strip of the circular elastically supported plate

Let us again consider a spherical shell circular on plan and subjected to a uniformly distributed load p. The edges are completely restrained, however. If the rise/chord ratio of the meridian section is not too large we can, for calculating the deformations w, replace the spherical shell by an elastically supported flat circular plate with a radius equal to that of the base of the shell (or, preferably, with a radius equal to the generating line of a cone tangential to the spherical shell at the base circle), this plate being completely restrained at its perimeter. Let r be the radius of the sphere and neglect ν. Then, according to (12), the primary foundation modulus will be:

$$c = \frac{2E\delta}{r^2} \quad \text{(32)}$$

Supposing this plate to be subjected to a load p, the displacements w and the shell moments can be found either analytically or experimentally.

We may ask ourselves whether it would, in this case too, not be sufficient merely to consider a diametral beam strip (see Fig. 9).

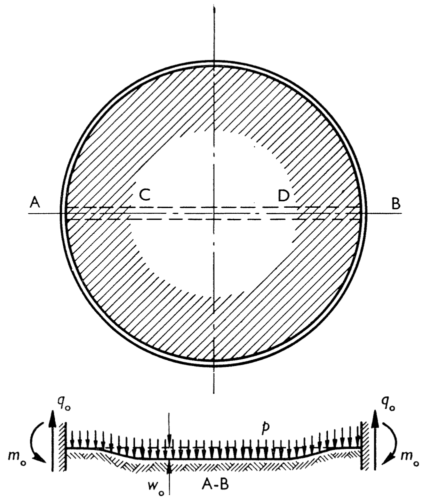


Fig. 9. Spherical shell, circular on plan, considered as an elastically supported plate. The edge is completely restrained. The shaded area is the edge disturbance zone. AB is a beam strip cut out of the plate. As part of the plate, however, the beam strip is subjected along its sides to shear forces and torsion moments (acting in the regions AC and BD) which have a relieving effect on the free beam.

Neglecting ν, we obtain – according to formule (24) – for the displacement of the spherical shell outside the zone of edge disturbance:

$$w_0 = \frac{pr^2}{2E\delta}.$$

The deformation equations for calculating the restraint reactions m_0 and q_0 of the elastically supported beam of unit width will then be:

$$\left.\begin{aligned} \frac{m_0\lambda}{EI} &= \frac{q_0\lambda^2}{2EI} \\ \frac{q_0\lambda^3}{2EI} - \frac{m_0\lambda^2}{2EI} &= w_0 \end{aligned}\right\} \quad \text{(33)}$$

Hence:

$$m_0 = \frac{2EIw_0}{\lambda^2} \quad \text{(34)}$$

$$q_0 = \frac{4EIw_0}{\lambda^3} \quad \text{(35)}$$

With formula (17) we obtain for the sphere:

$$\lambda_b{}^2 = \frac{\delta r}{\sqrt{3}\sqrt{2}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (36)$$

and for the circular cylinder with radius r:

$$\lambda_c{}^2 = \frac{\delta r}{\sqrt{3}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (37)$$

Neglecting the deformation load, we must, for calculating the restraint moment m_0, substitute $\lambda_b{}^2$ according to (36) for λ^2 in formula (34).

It should be noted, however, that by considering a detached diametral beam strip too unfavourable a distribution of forces is found, because in the edge disturbance zones AC and BD, as part of the complete circular plate, shear forces and torsion moments are actually present which have a relieving effect.[1])

Consequently the calculated moments will be larger than those occurring in reality. A further reduction will in this case be caused by the deformation load.

On substituting for λ^2 in formula (34) the value $\lambda_c{}^2$ according to (37), which relates to a cylinder, we obtain the restraint moment according to GECKELER's approximate theory, which is, however, smaller than the correct value.[2]) Hence, if the deformation load is neglected, it is safer to apply the characteristic length based on the primary foundation modulus of the sphere. The fact the characteristic length of the cylinder of revolution occurs in GECKELER's approximation can be explained with the aid of the deformation load. Because of axial symmetry, we have:

$$\frac{\partial v}{\partial y} = 0, \text{ hence } \bar{p} = \frac{E\delta}{r}\frac{\partial u}{\partial x} = \frac{E\delta}{r}\left[\frac{n_x}{E\delta} + \frac{w}{r}\right] = \frac{n_x}{r} + \frac{E\delta w}{r^2} \quad . \; . \quad (38)$$

For $\frac{E\delta}{r^2} = {}^1/_2\, c$ (where c is the primary foundation modulus for the sphere) and putting $\frac{n_x}{r} = -{}^1/_2 p$, which is an acceptable assumption if the central angle of the spherical shell is not too small, the differential equation for a meridian strip will be:

$$K\frac{d^4 w}{dx^4} = p - {}^1/_2 p + {}^1/_2 cw - cw$$

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1) In this connection it is appropriate to recall that, for example, in a circular plate uniformly loaded over its whole area (freely supported or completely restrained at the perimeter), the moments in a diametral beam strip, as part of the plate, are only $^3/_8$ of those in the detached beam subjected to the same load.

2) Cf. TIMOSHENKO, S. Theory of Plates and Shells, McGraw-Hill, New York, 1940, pp. 472–475. See also HETÉNYI, M. Spherical Shells subjected to Symmetrical Bending, Publ. Int. Ass. for Bridge and Struct. Eng., Vol 5, 1937/1938, pp. 173–175.

or: $$K \frac{d^4w}{dx^4} = {}^1/_2 p - {}^1/_2 cw \quad (39)$$

We see that the particular solution of this equation remains $w_0 = \frac{p}{c} = \frac{pr^2}{2E\delta}$ whereas the value of the foundation modulus has become half that for the sphere, i.e., equal to that for the cylinder of revolution.

In his plate analogy for the feebly curved shell WLASSOW also arrives at a foundation modulus equal to that for the cylinder of revolution, this evidently being – as in GECKELER's method of analysis – correct only for axially symmetrical loading.

It should, however, be borne in mind that the load which has to be applied to the analogous plate or beam may differ considerably from the actual load on the shell.

In more general terms it can be said that, for a shell of revolution with axially symmetrical loading and a meridian section of arbitrary shape the secondary foundation modulus will be $\frac{E\delta}{r_2^2}$, where r_2 is the second principal radius of curvature (length measured along the normal up to its intersection with the axis of revolution).

For $\frac{\partial v}{\partial y} = 0$ and neglecting ν, we obtain from (11) and (5):

$$\bar{p}_z = E\delta k_1 \frac{\partial u}{\partial x} = \frac{E\delta}{r_1}\left(\frac{n_1}{E\delta} + \frac{w}{r_1}\right)$$

which in combination with (10) and (12) gives:

$$K\Delta\Delta w = p_z + \frac{n_1}{r_1} - \frac{E\delta}{r_2^2} w \quad (40)$$

If ν is not neglected, we obtain:

$$K\Delta\Delta w = p_z + \left(\frac{1}{r_1} + \frac{\nu}{r_2}\right) n_1 - \frac{E\delta}{r_2^2} w \quad (41)$$

5 Conclusions

The conception of the doubly curved shell as an elastically supported plate has much to commend it. In highly complex cases it would, for the present, appear to be the only method available to the design engineer for estimating the approximate magnitude of the edge disturbance moments. It should be borne in mind, however, that a doubly curved shell can reasonably be replaced by a flat plate only if we confine ourselves to a so-called quasi-Euclidean region, i.e., a region around the origin of the chosen system of co-ordinates where the principles of plane geometry are still reasonably valid. This can be assumed to be so up to a rise/chord ratio of about $^1/_7$. For calculating the primary foundation modulus at any particular point both the principal radii of curvature must be taken into account.

This foundation modulus is valid only if the deformation load is zero or negligible. If the deformation load is taken into consideration, then the actual load on the analogous plate or beam as well as the primary foundation modulus will, generally speaking, be affected. This gives rise to a secondary foundation modulus. In shells of revolution with axially symmetrical loading this secondary modulus is dependent only on the second principal radius of curvature, this being in agreement with the foundation modulus as applied by GECKELER and WLASSOW. In the case of a cylindrical barrel vault uniformly loaded over its entire surface, presenting an elongated rectangular shape on plan and supported on all four sides, the secondary foundation modulus may, for a shell strip extending in the transverse direction, even become zero.

The analysis as a plate, or a beam, supported on an elastic foundation appears to be capable of yielding useful results in dealing with spherical shells, hyperbolic paraboloid shells bounded by generating lines, and elliptic paraboloid shells rectangular on plan.[1])

It will have to be further investigated whether this analysis can be applied to other cases and also whether it can be applied if the deformation load is neglected but suitable load factors are introduced at the same time.

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[1]) GIRKMANN, K. Flächentragwerke, 5th ed., Springer-Verlag, Vienna, 1959, pp. 462–464. See also the publication mentioned in note 1 on p. 229 of that book.