HERON
ENGLISH EDITION No. 3 (1965)

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STABILITY AND PLASTIC DESIGN (1)

U.D.C. 624.042

The ultimate load P_p of a structure is usually calculated with the aid of the elementary plastic analysis. Numerous experiments have shown that this method of analysis generally yields reliable results. In the case of very slender structures, however, the stability may adversely affect the ultimate load.
In order to estimate this effect, an eccentrically loaded member is considered in the present paper. It appears that the actual collapse load P_{kr} can be found with the aid of a simple formula in which the ultimate load P_p, calculated according to the elementary plastic analysis, and the Euler buckling load P_E play a part.
A corresponding formula is found for the calculation of the actual collapse load of any particular portal frame whose displacements are confined to its own plane. In order to determine for such a portal frame the load corresponding to P_E, a caricature model of the structure is used which is much more slender than the latter and which can be constructed in a very simple manner from small strips of mild steel.
The method presented may be of importance, not only for arriving at more exact design rules for normal structures, but also for collapse analysis for cases such as fire or war damage, under which conditions the loading may approach the collapse load very closely.

0 Introduction

The object of plastic design, or collapse analysis, is to design a structure for ultimate strength, i.e., it is endeavoured to determine the load at which the structure actually collapses. In general, and more particularly with regard to steel structures, this leads to a simple design method, as it can be assumed that a section, on reaching a certain bending moment M_p, will develop a "plastic hinge" which always transmits the same moment, whatever the magnitude of the angular rotation. With increasing load the structure will collapse when so many plastic hinges have developed that it becomes a "mechanism", i.e., becomes unstable. No further increase of the load is possible.

By means of plastic design it is possible to obtain a good idea of the reserve strength (load capacity) that the structure still has available when the permissible load according to the elastic methods of analysis has been reached. An obvious conclusion is that substantial savings in materials can be effected in certain cases. For this reason many of the structures hitherto erected, designed in accordance with the plastic theory, convey an impression of slenderness. A number of examples of such structures are to be found, inter alia, in "The Steel Skeleton", Vol. II, by BAKER, HEYMAN and HORNE. The question that may be asked is whether one can perhaps go too far in this direction, so that, for example, there arises a risk that a structure becomes so slender as to endanger its stability. Can a reliable criterion for this be established?

The "exact" theory of buckling in the elasto-plastic range is particularly complex. To find a method whereby the stability can be judged and which ties up tolerably well with the elementary methods of plastic design, it will be necessary to introduce major simplifications in order to obtain a sufficiently convenient and manageable result.

1 Eccentrically loaded compression member

In order to gain some insight into the problem, the simple case of a straight eccentrically loaded compression member – a prismatic bar – will be considered (see Fig. 1). This member has a length l, a stiffness EI and a plastic moment M_p. All its deformations are assumed to be confined to the plane of drawing. On applying the elementary plastic analysis to this member, we should find:

$$M_p = P_p\, e \qquad (1)$$

which means that the member behaves elastically up to the load $P_p = M_p/e$.

When the load P_p is reached, a plastic hinge develops over the entire length of the member, and the ultimate load capacity is attained. If the member has a finite stiffness EI, however, deflections occur in consequence of the bending moment. The bending moment that actually occurs in the middle of the member increases, namely, $M = Py$ (see Fig. 1). If the deflection y were caused by the moment $M = Pe$, we should find:

$$y = e + \frac{P\,e\,l^2}{8EI} \qquad (2)$$

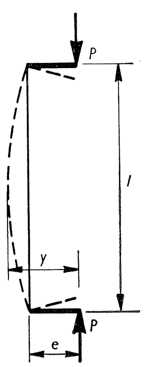


Fig. 1. Eccentrically loaded member.

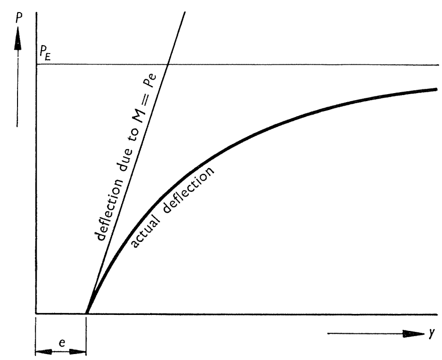


Fig. 2. Relation between the deflection and load of an eccentrically loaded member.

In reality, however, the moment is not equal to Pe, but to Py, so that the deflections become larger.

It is known that the following approximation is reasonably accurate:

$$y = \frac{n}{n-1}\, e \qquad \text{(3a)}$$

where n denotes the ratio between the buckling load P_E according to EULER and the load P, actually applied:

$$n = \frac{P_E}{P} \quad \text{and} \quad P_E = \frac{\pi^2 EI}{l^2}.$$

Hence we can write:

$$y = \frac{e}{1 - \dfrac{P}{P_E}} \qquad \text{(3b)}$$

The member will collapse when $P = P_{kr}$, so that $P_{kr}\, y = M_p$. Therefore:

$$M_p = P_p\, e = \frac{P_{kr}\, e}{1 - \dfrac{P_{kr}}{P_E}} \qquad \text{(4a)}$$

and hence:

$$P_{kr} = \frac{P_E\, P_p}{P_E + P_p} \qquad \text{(4b)}$$

or:

$$\frac{1}{P_{kr}} = \frac{1}{P_E} + \frac{1}{P_p} \qquad \text{(4c)}$$

This simple formula is found to provide a very good means of predicting the collapse load of an eccentrically loaded member. For members with not too short and stout dimensions the effect of the axial thrust upon the plastic moment is not great and can therefore usually be neglected.

2 Framework of members

On passing from this simple member to a more complex system of members – e.g., a portal frame – it is obvious to suppose that the elastic stability and the ultimate load according to the elementary plastic analysis will play a part with regard to the strength of such a structure as well.

Under a load of small magnitude this system will display an entirely elastic behaviour. The deflections z_0 can be deduced from the distribution of forces as determined for the undeformed system (the ordinary elastic analysis). It will be assumed that no displacements other than in the plane of drawing can occur. As the deformations will also have some influence upon the distribution

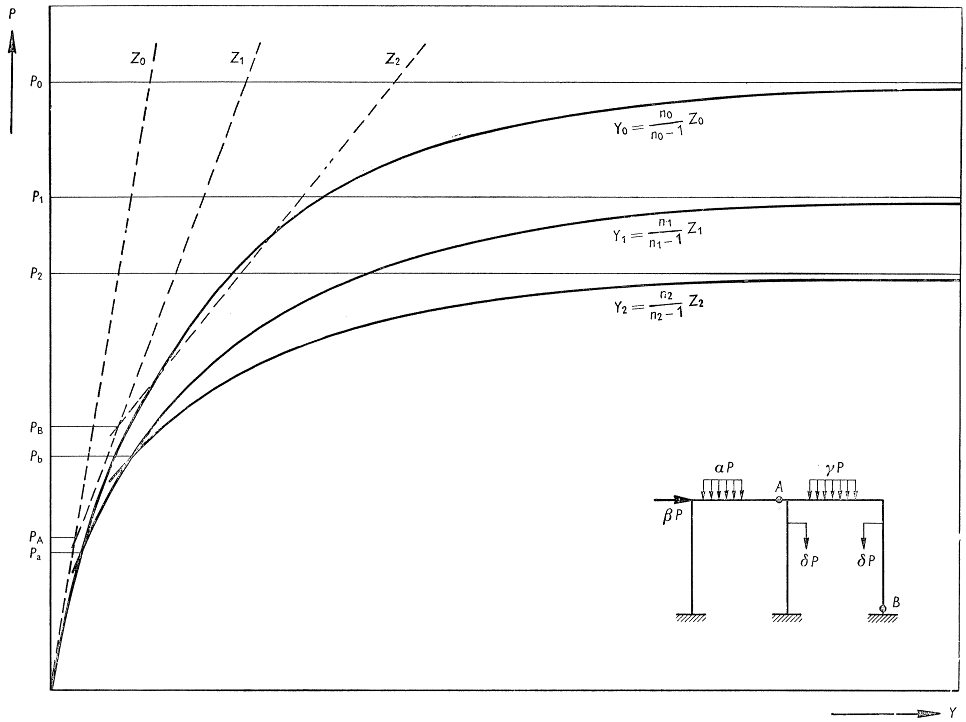


Fig. 3. Relation between the displacement y of an arbitrary point of a structure (e.g., the horizontal displacement of the horizontal member of the system indicated) and the load. The following quantities are involved:
the elastic buckling load P_0: before any plastic hinges have developed
P_1: for the structure with a hinge at A
P_2: for the structure with hinges at A and B
According to an ordinary elastic analysis – i.e., neglecting the effect of deformations – a plastic hinge at A would be formed at a load $P = P_A$. Actually, this occurs at a somewhat lower load P_a because the displacements y_0 also have some effect on the distribution of forces in the structure. After the plastic hinge at A has developed, the structure is more flexible, so that displacements y_1 occur. For $P = P_b$ the next hinge is formed, at B, so that the system becomes even more flexible and displacements y_2 occur, etc.

of forces, they will be multiplied by a small magnification factor $\frac{n_0}{n_0-1}$, where $n_0 = P/P_0$ and P_0 is the elastic buckling load of the portal frame conceived as entirely elastic. The actual deformations are then (see Fig. 3):

$$y_0 = \frac{n_0}{n_0-1}\, z_0 \quad (5)$$

When P has attained the value P_a, then the moment at point A has become equal to M_p, so that a plastic hinge develops there. Even now the portal frame still behaves elastically. Only at the plastic hinge is there a known "external" moment acting upon the structure, but otherwise the hinge behaves like an ordinary hinge. A fresh distribution of forces can now therefore be determined, whence – in this case with a magnification factor $\frac{n_1}{n_1-1}$ – the deflections y_1 can be deduced. Here $n_1 = P_1/P$ and P_1 is the elastic buckling load of the portal frame with a hinge at A. Naturally P_1 is smaller than P_0. Continuing in this way, for $P = P_b$ another hinge is formed, this time at B (see Fig. 3). Let P_2 denote the elastic buckling load of the portal frame with two hinges, while $P_2 < P_1 < P_0$.

In this manner a number of hinges are formed until a system with n hinges is obtained which has an elastic buckling load P_N which is smaller than the force P_n at which the hinge n was formed. The most obvious case is the one considered by the elementary plastic analysis, namely, where $P_N = 0$, when the structure with n plastic hinges has become a mechanism. However, in slender structures the load capacity may in some cases decrease to such an extent already with a smaller number of plastic hinges that collapse occurs.

Assuming that the place N where the nth and last plastic hinge will shortly develop is known, it is clear that in the system with $n-1$ hinges the moment at N is characterised by the force R_{n-1} which this section has to transmit and by the eccentricity z_{n-1} of this force. If the force distribution is represented by a line of thrust along which certain force components act, then this distribution can be determined for the undeformed system with $n-1$ hinges. The force R_{n-1} is then linearly dependent upon P, and the eccentricity z_{n-1} is known. Because of the deformations, the eccentricity will approximately be multiplied by a factor $\frac{n_{n-1}}{n_{n-1}-1}$, while the magnitude of R_{n-1} will undergo relatively little change, inasmuch as R_{n-1} is obtained by resolution of the externally applied load system.

The plastic hinge at N will therefore develop when:

$$R_{n-1} \frac{n_{n-1}}{n_{n-1}-1} z_{n-1} = M_p \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (6)$$

If a straight member (as discussed on page 3) is considered for comparison, this member can be made as far as possible equivalent to the given structure. This means that for this member, loaded by a force P with eccentricity e_{n-1}, the following is valid:

Euler buckling load P_E = elastic buckling load P_{n-1} of the portal frame with $n-1$ hinges;

critical load $P_{kr} = P_{kr}$ of the portal frame;

ultimate load according to elementary analysis $P_p = P_p$ of the portal frame.

As R_{n-1} is linearly dependent upon P, we can write:

$$P\, e_{n-1} = R_{n-1}\, z_{n-1} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (7a)$$

At the instant when the nth and last hinge is formed, P becomes equal to P_{kr}, so that:

$$P_{kr}\, e_{n-1} = R_{n-1}\, z_{n-1} \qquad (7b)$$

or, having regard to (6):

$$P_{kr}\, e_{n-1} \frac{n_{n-1}}{n_{n-1}-1} = R_{n-1}\, z_{n-1} \frac{n_{n-1}}{n_{n-1}-1} = M_p \qquad (7c)$$

From the elementary plastic analysis – i.e., neglecting the deformations – it is known that:

$$P_p\, e_{n-1} = M_p \qquad (8)$$

at any rate, if it is assumed that in both cases it is the same hinge that is the last to develop. Hence it can be stated that:

$$P_p\, e_{n-1} = P_{kr}\, e_{n-1} \frac{n_{n-1}}{n_{n-1}-1} \qquad (9a)$$

or:

$$P_{kr} = \frac{n_{n-1}-1}{n_{n-1}} P_p = \frac{P_{n-1}-P_{kr}}{P_{n-1}} P_p \qquad (9b)$$

so that:

$$\frac{1}{P_{kr}} = \frac{1}{P_p} + \frac{1}{P_{n-1}} \qquad (9c)$$

It is therefore not unreasonable to assume that the actual collapse load P_{kr} of a portal frame structure, like that of an eccentrically loaded compression member, can very simply be expressed in:

$P_p =$ the ultimate load of the structure according to the elementary plastic analysis;

$P_{n-1} =$ the elastic buckling load of the – in many cases statically determinate – system in which all the hinges except the last have developed.

It is not, however, a simple matter to determine P_{n-1} by calculation because, before the penultimate plastic hinge occurs, the stability may already have an appreciable effect on the distribution of forces. For this reason the elementary plastic analysis is not always able to predict with sufficient certainty where the plastic hinges will occur and in what order they will develop.

3 Using a model as an aid

An obvious idea is to make use of a model for solving this problem. Of course a model which is a "caricature" of the portal structure to be investigated, and in which the effect of stability plays an exaggerated part, has many advantages over an entirely realistic model. Because of the great slenderness of such a "caricature" model, the collapse load can be expected to be considerably lower than that predicted by the elementary plastic analysis. For the actual structure we can put: $EI/l^2 = A$ and $M_p/l = K$. For the model these values are A_1 and K_1 respectively.

The ultimate load according to the elementary plastic analysis is $P_p = pK$ for the actual structure and $P_p' = pK_1$ for the model. Furthermore, if the plastic hinges occur in the same sequence and in the same places both in the model and in the actual structure, then the buckling load P'_{n-1} of the model can be expressed in terms of the buckling load of the actual structure, as follows:

$$P'_{n-1} = \frac{A_1}{A} P_{n-1} \quad \ldots \quad (10)$$

In reality, P'_{n-1} will usually be a little smaller, since the model dimensions are so chosen that the deformations in the model have more effect upon the formation of the plastic hinges than they have in the actual structure.

The collapse load of the model P'_{kr} can be expressed in P'_p and P'_{n-1}, according to (9c):

$$\frac{1}{P'_{kr}} = \frac{1}{P'_p} + \frac{1}{P'_{n-1}} \quad \ldots \quad (11a)$$

whence it follows that:

$$\frac{1}{P'_{n-1}} = \frac{1}{P'_{kr}} - \frac{1}{P'_p} \quad \ldots \quad (11b)$$

The buckling load of the actual structure can then, with the aid of (10), be determined as follows:

$$\frac{1}{P_{n-1}} = \frac{A_1}{A} \frac{1}{P'_{n-1}} = \frac{A_1}{A}\left(\frac{1}{P'_{kr}} - \frac{1}{pK_1}\right) \quad \ldots \quad (12)$$

This provides a somewhat low estimate of P_{n-1}, which is on the safe side. The collapse load P_{kr} of the actual structure can now be predicted by means of the surprisingly simple formula obtained by substitution of (12) into (9c):

$$\frac{1}{P_{kr}} = \frac{A_1}{A}\left(\frac{1}{P'_{kr}} - \frac{1}{pK_1}\right) + \frac{1}{pK} \quad \ldots \quad (13)$$

In this formula A and K are quantities derived from the actual structure. A_1 and K_1 are determined by the dimensions of the model, while p is found with the aid of the elementary plastic analysis. P'_{kr} is the collapse load of the model. The fact that only the collapse load has to be determined makes the model technique particularly simple.

4 Model technique

A model can be constructed from strips of mild steel. The steel should have a definite yield point. M_p and EI can be determined with the aid of a simply-supported beam or a cantilever. The value of EI can, however, also be determined with sufficient accuracy by calculation. For all the members it is necessary to assign a certain constant value both to the ratio A_1/A and to the ratio K_1/K. Of course, A_1/A will have to be considerably smaller than K_1/K because $a = A/K = \dfrac{EI}{l^2} : \dfrac{M_p}{l}$ is a measure of the effect of the stability upon the load capacity. According as a is smaller, the structure will be more flexible, and the model will have to be exaggeratedly flexible. Since $M_p = {}^1/_4 bh^2\sigma_v$ and $I = bh^3/12$, the ratio $a = EI/M_pl = Eh/3\sigma_v l$ can be given any desired value by suitably choosing h, i.e., the thickness of the steel plate from which the strips are cut. Also, the values of M_p and EI can be chosen as desired, by suitably selecting the width b for the strips.

The model can be assembled very simply with the aid of small clips or Meccano components or even by welding. A great advantage is that, generally speaking, the lines of thrust of the forces are far away from the centrelines of the members. For this reason initial curvatures of the members, which in more theoretical cases of buckling always present numerous experimental difficulties, are of no consequence in the models considered here. Nor has the precise shape of the moment-curvature diagram much effect on the results. So long as M_p does not change, the strength of the model remains practically unchanged, so that in many cases it is possible to straighten out the model after it has collapsed and to re-use it, for example, for the investigation of a different loading condition.

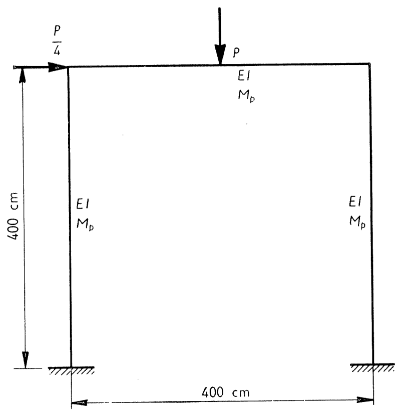


Fig. 4.

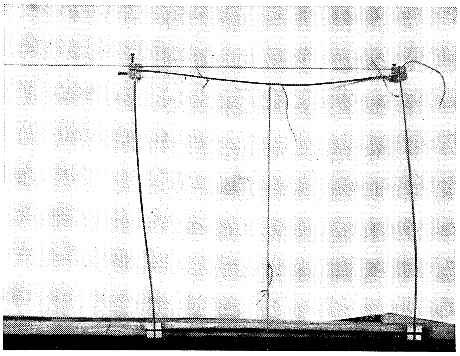
By way of illustration, an example of a portal frame constructed of DIN 10 rolled steel sections (as represented in Fig. 4) will be considered. It is assumed that the structure is entirely restrained against lateral displacement out of its own plane. If we adopt for the value of a in the model one-tenth of the corresponding value for the actual structure, then the slenderness of the model will be considerably exaggerated.

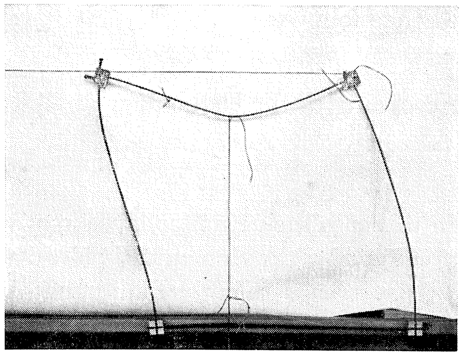
The following values relate to the structure:

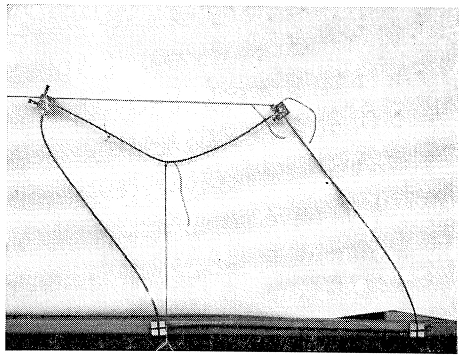
$$A = \frac{EI}{l^2} = 6450 \text{ kg} \quad \text{and} \quad K = \frac{M_p}{l} = 655 \text{ kg}$$

so that:

$$a = \frac{A}{K} = \frac{6450}{655} = 9.82$$







Photographs 1, 2 and 3
Various stages of deformation of a model of a portal frame under increasing load. The displacements occurred in the plane of the structure only.

The model was constructed from mild steel strips with cross-sectional dimensions of 1.5 mm $\times$ 10 mm. Hence it follows that $EI = 605$ kgcm^2 and $M_p = 13$ kgcm, determined with the aid of a three-point bending test. Therefore we have for the model:

$$a = \frac{EI}{M_p\, l} = 0.1 \times 9.82 = 0.982$$

so that:

$$l = \frac{605}{0.982 \times 13} = 47.4 \text{ cm}$$

The values for A_1 and K_1 could now be calculated. These were:

$$A_1 = \frac{EI}{l^2} = 0.27 \text{ kg} \quad \text{and} \quad K_1 = \frac{M_p}{l} = 0.275 \text{ kg}$$

According to the elementary plastic analysis the ultimate load is $P_p = 8M_p/l$ (see photographs 1, 2 and 3). However, the model collapsed at $P'_{kr} = 1.8$ kg $= 6.55 M_p/l$.

The collapse load of the actual structure could now be determined from (13):

$$\frac{1}{P_{kr}} = \frac{0.27}{6450}\left(\frac{1}{1.8} - \frac{1}{8 \times 0.275}\right) + \frac{1}{8 \times 655} = 195 \times 10^{-6} \text{ kg}^{-1},$$

whence follows:

$$P_{kr} = 5130 \text{ kg}$$

According to the elementary analysis the ultimate load would be:

$$P_p = 8\frac{M_p}{l} = 8 \times 655 = 5240 \text{ kg}$$

In this case the actual collapse load was therefore 2.3% smaller than the value predicted by the elementary plastic analysis.

It was subsequently found that a calculation for this case can also yield fairly good results. To this end, a few plausible assumptions must be made with regard to the nature of the deformations that occur. A calculation of this kind, however, will always be very laborious, as it is also necessary to make assumptions as to the location of the last hinge and as to the number of hinges at which collapse will occur.

5 Three-dimensional case

So far, we have proceeded on the assumption that the structure can undergo displacements only in its own plane. The more general case, where a section will not necessarily remain within the plane of the structure (see Fig. 5), is of course much more complex.

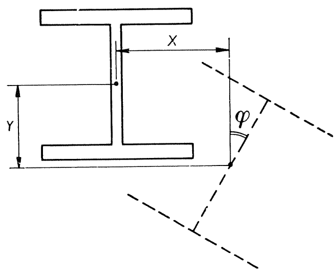
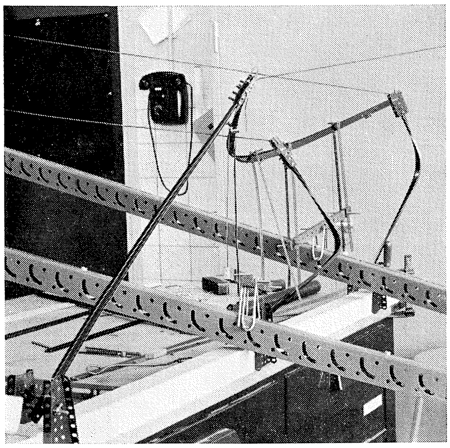


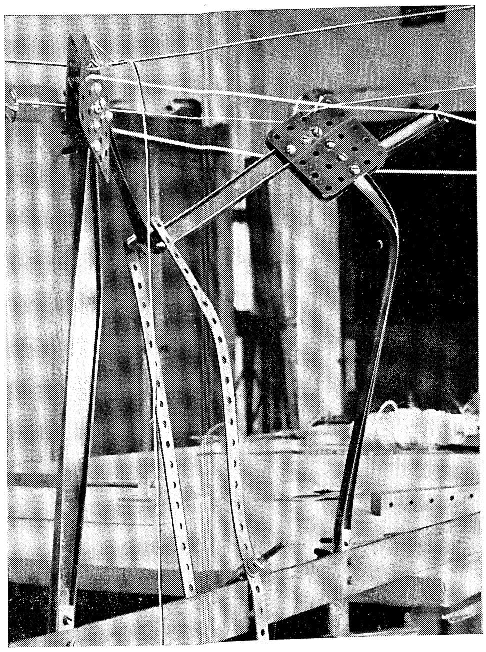
Fig. 5.

Three kinds of displacements are possible, namely x, y and φ, each of which is suggestive of certain cases of buckling. With y is evidently associated the occurrence of buckling within the plane of the system, and with x and φ is associated the occurrence of buckling in a direction perpendicular to this plane, e.g., lateral buckling and lateral-torsional buckling. There are indications that, also for the three-dimensional case, it is possible to investigate the effect of stability upon the load capacity with just as simple means as those which have been described for the two-dimensional case. This is important because most structures which are loaded approximately up to collapse do indeed function as three-dimensional structures – at any rate, if all the component parts have been designed for approximately the same loads. To investigate a case of this kind, three different "caricature" models would be needed. What curious collapse shapes may develop are shown by photographs 4 and 5. The models illustrated are made from curtain-rail sections.

6 Conclusion

In normal cases the collapse load can be predicted with reasonable accuracy





Photographs 4 and 5
Collapse shapes of some models of frameworks in which the displacements were not confined to the plane of the structure.

with the aid of the elementary plastic analysis. It is of course advisable to ensure the stability of the structure as far as possible. In the well known book entitled "The Steel Skeleton", Vol. II, by BAKER, HEYMAN and HORNE, detailed consideration is therefore given to the method of so choosing minimum dimensions of columns and beams that the calculated load capacity will not be adversely affected by phenomena of buckling and lateral-torsional buckling.

In certain circumstances, however, it may be of importance to determine the collapse load of a very slender structure. For example, this case may arise if it is felt to be desirable to establish design rules of greater accuracy than those at present employed by BAKER and his associates. Furthermore, it may occur that the purpose for which an existing structure is used is changed, so that an analysis based on the collapse load becomes necessary. Important applications are conceivable in the case of structures which play a vital role in times of war, such as factories and air-raid shelters, for which the actual collapse load, even when columns have been destroyed by bomb explosions, may be of importance. This also applies to outbreaks of fire in which the rise in temperature causes a lowering of the yield point and the modulus of elasticity of steel. In addition, the increase of creep will make the deformations increase in magnitude. Damage and the danger to human life will be very much less if the structure nevertheless does not collapse.

In such cases the method discussed here – which can probably also be extended to three-dimensional structures – provides a simple means of estimating whether the actual collapse load is perhaps lower than is predicted by the elementary plastic analysis. The requisite model, which constitutes an exaggeratedly slender representation of the structure, is very simple to construct and to test.