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THE THEORY OF THE COUPLED SPRING FOUNDATION AS APPLIED TO THE INVESTIGATION OF STRUCTURES SUPPORTED ON SOIL

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The concept of elastic foundation as a number of springs functioning independently of one another is not quite correct in a good many cases. To overcome this drawback, various authors have introduced a simple form of interaction of the springs. The foundation model obtained in this way – called the "coupled spring foundation" – is characterised by a number of features which are also encountered in actual foundations. This is, inter alia, also found to be the case with structures supported on soil.
The theory is applied to model research on airfield runways. The interpretation of field tests for determining the foundation parameters is discussed.

0 Introduction

For the analysis of structures which are supported at a large number of points or which rest entirely on a bearing surface the concept of "elastic foundation" is often employed. In general this results in a convenient and simple analysis [1]. However, a number of idealised assumptions, as distinct from the actual conditions, are accepted. Thus, for example, the time effect and the non-linear load-deflection relationship of soil are ignored. In the present paper attention is focussed on the following aspect: when a local load is applied to the base, the surroundings of the loaded area can also co-operate in carrying the load.

For certain structures this effect involves a considerable modification of the distribution of forces, so that the simple model of independently functioning springs does not provide a sufficiently accurate basis for the analysis.

This is more particularly true of the analysis of rails, inasmuch as the "counterpressure" exerted by a sleeper depends on the deflection of adjacent sleepers. In order to describe this coupling effect, van der Eb and de Pater have derived a differential equation which gives a good account of the actual behaviour of the foundation, as has been confirmed by tests [2].

The differential equation for a beam or slab supported on a coupled spring foundation is found to be the same as that for a structure on an ordinary elastic foundation if a constant tensile normal force is acting. A base comprising springs which resist rotation of the surface is also described in terms of a coupled spring foundation by the same differential equation. A result of this analogy is that a number of solutions are already available, both for beams [1] and for slabs [3].

In the present paper the application of the theory to model research on airfield runway pavements will be described. The investigation of the foundation of the pavement showed the coefficient of soil reaction to be dependent on the size of the loaded area. The same phenomenon was observed to occur in the foundation material of the reduced-scale model. To deduce the law of model similarity it then becomes necessary to take the coupling effect into account.

This paper is arranged as follows:

First, the analysis for a number of standard cases is given. In this analysis the similarity of behaviour between a foundation on soil and a coupled spring foundation is already manifested.

Next, the plate bearing test is dealt with, which is used for determining the properties of the soil in situ. The method of determining the parameters of the coupled spring foundation from the test results is discussed.

Finally, it will be shown that the theory adequately accounts for the behaviour of the soil and of the model material. Hence it is reasonable to suppose that the theory can also be successfully applied to the investigation of other structures supported on soil.

Notation

x, y	co-ordinates of a point of the supporting surface
x	also: radius in polar co-ordinates; differentiation with respect to x is indicated by a prime
p	area loading (kl^{-2})
w	settlement of the supporting foundation (l)
k	true modulus of subgrade reaction, i.e., resistance per unit settlement per unit area (kl^{-3}) if the same settlement occurs over the entire site
k^*	apparent modulus of subgrade reaction, due to co-operation of the material beside a local settlement (kl^{-3})
f	ratio between true and apparent modulus of subgrade reaction $k : k^*$ (1)
T	shear force per unit width between two foundation elements (kl^{-1})
A	shearing constant (kl^{-1})
b	co-operating width, derived from A (l)
s	reciprocal value of b (l^{-1})
F	area of a bearing plate (l^2)
I_0, I_1, K_0, K_1, K_2	Bessel functions
Δ	Laplacian operator: $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

1 Theoretical consideration of the coupling effect

1.1 Derivation of the differential equation

The "foundation" is composed of spring elements which are coupled to one another. First, a plane state of strain will be considered. The coupling between two elements transmits a shear force T per unit width (i.e., width perpendicular to the plane of the drawing). This force is associated with the difference in deflection between the elements. A simple assumption is that the shear force is proportional to the difference in deflection between two consecutive elements and therefore to the first derivative of the deflection:

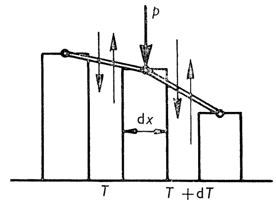

Fig. 1.

$$T = Aw' \quad (1)$$

The spring element has its own spring constant k.
The equilibrium of an element is expressed by the equation:

$$p \cdot dx - kw \cdot dx - T + (T + dT) = 0$$

Hence it follows, in combination with (1)

$$-Aw'' + kw = p \quad (2)$$

In subsequent formulae the shear constant A would occur under the root sign. In order to enable these formulae to be written in a simpler form, new quantities b and s are introduced, for which:

$$b^2 = \frac{A}{k} \quad \text{and} \quad s = \frac{1}{b} \quad (3)$$

The quantity b has the dimension of a length and is called the "co-operating width" for reasons which will presently be apparent. With the reciprocal value s the differential equation (2) can be written as:

$$-w'' + s^2 w = \left(\frac{p}{k}\right) s^2 \quad (4)$$

with the solution:

$$w = \frac{p}{k} + W_1\, e^{-sx} + W_2\, e^{+sx} \quad (5)$$

The loading cases to be investigated have the character of a local disturbance in otherwise undisturbed surroundings. In the solution for large positive values of x the term with the indeterminate coefficient W_1 represents the decreasing deflection due to the influence of the local disturbance. The term with W_2 does the same for large negative values of x (the solution will therefore always comprise two different branches).

Various basic cases for the plane state of strain will be discussed in Section 1.2.

In the general case, where the deflection can vary both in the X- and in the Y-direction, the term containing A in the differential equation (2) must comprise the sum of the two derivatives in the X- and Y-direction:

$$-A\Delta w + kw = p \quad \text{. . . . . (6)}$$

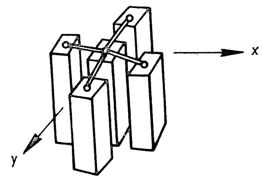

Fig. 2.

In the following, axially symmetric cases will be considered, in which only one independent variable remains.

On considering the equilibrium of a basic element in the shape of a hollow cylinder with a wall thickness dx, we find:

$$(p-kw)2\pi x\mathrm{d}x - T\cdot 2\pi x + (T+\mathrm{d}T)2\pi(x+\mathrm{d}x) = 0$$

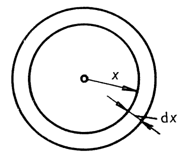

Fig. 3.

In combination with (1) this gives:

$$-A\left(w'' + \frac{1}{x}w'\right) + kw = p \quad \text{. . . . . (7)}$$

This equation also follows directly from (6) by conversion to the axially symmetric form.

A further simplification is obtained by the introduction of s:

$$-\left(w'' + \frac{1}{x}w'\right) + s^2 w = \left(\frac{p}{k}\right) s^2 \quad \text{. . . . . (8)}$$

with the solution:

$$w = \frac{p}{k} + W_1\cdot K_0(sx) + W_2\cdot I_0(sx) \quad \text{. . . . . (9)}$$

where K_0 and I_0 are Bessel functions. A number of axially symmetric cases have been worked out in Section 1.3.

1.2 Solutions for a plane state of strain

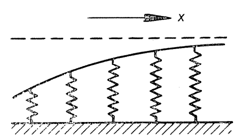

Fig. 4.

For areas carrying no load the damping solution for positive x is:

$$w = W_1 e^{-sx}$$

The coupling transmits a shear force, which can be calculated from (1)

$$T = -AW_1 se^{-sx} = -kbW_1 e^{-sx} = -kbw$$

The minus sign indicates that each spring is pulled downward by the coupling on the left-hand side (x is reckoned as positive to the right). The shear force transmitted by each coupling can also be determined by considering that this resistance is indirectly developed by the springs to the right of the section under investigation:

$$T = \int_x^\infty kw\mathrm{d}x = -k\frac{1}{s}W_1 e^{-sx} = -kbw$$

In this particular case the shear force is everywhere proportional to the deflection (as is the slope). The constant of proportionality is kb.

Basic case a

Loading by a line load

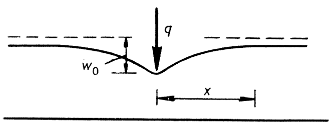

Fig. 5.

The load q is uniformly distributed over a line perpendicular to the plane of the drawing. The loaded width within that plane is zero. This means that, for a finite deflection w_0, the spring resistance under the load is an order of magnitude smaller than the load. All the bearing capacity must therefore be provided by the adjacent material. On each side the deflection presents a shape corresponding to the damping branch of the solution; both sides provide a shear force kbw_0.

From $q = 2kbw_0$

$$\left.\begin{aligned} &\text{it follows that} \quad w_0 = \frac{q}{2kb} \\ &\text{and on the right} \quad w = \frac{q}{2kb}\, e^{-sx} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (10)$$

If the subgrade consisted of non-coupled springs, the load q, spread over a width $2b$, would produce the same deflection w_0. This explains the term "co-operating width" which has been applied to b.

Basic case b

Distributed load over a strip 2l

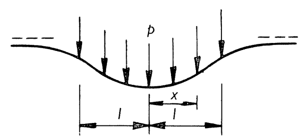

Fig. 6.

The solution for this loading case can be obtained by directly solving the differential equation or, alternatively, by integration of the previous case. The latter method will be employed here.

For a point $x_0 > l$ the contribution made to the deflection by a load element $p\,\mathrm{d}x$ is:

$$\frac{p\,\mathrm{d}x}{2kb}\, e^{-s(x_0-x)} = \frac{p}{2k}\, e^{-sx_0}\, \mathrm{d}e^{sx}$$

Integration over the loaded strip gives:

$$w = \frac{p}{2k}\, e^{-sx_0}(e^{sl} - e^{-sl}) = \frac{p}{k}\,(\sinh sl)e^{-sx_0}$$

In particular for the edge point $x_0 = l$ we have:

$$w = \frac{p}{2k}(1 - e^{-2sl}) \quad \ldots\ldots\ldots\ldots\ldots \quad (11)$$

A point of the loaded strip, for which $-l < x_0 < l$ can be regarded as the edge point of two loaded strips having a width $(l + x_0)$ on the left and $(l - x_0)$ on the right.

Application of formula (11), taking account of the strip width, gives the deflection:

$$w = \frac{p}{2k}[\{1 - e^{-s(l+x_0)}\} + \{1 - e^{-s(l-x_0)}\}]$$

$$= \frac{p}{2k}[2 - e^{-sl}(e^{sx_0} + e^{-sx_0})] = \frac{p}{k}(1 - e^{-sl}\cosh sx_0)$$

In solving directly from the differential equation it is necessary to satisfy the boundary condition that at the edge point the slopes under the loaded and the unloaded part are equal. The solution obtained is seen to satisfy this condition, since for both parts the following holds true:

$$\text{if } x = l, \text{ then } w' = -\frac{ps}{k} e^{-sl} \sinh sl$$

The deflection at the centre is also of particular interest. It is:

$$w = \frac{p}{k}(1 - e^{-sl}) \quad \ldots\ldots\ldots\ldots\ldots \quad (12)$$

On comparing the deflection at the centre and at the edge, it appears that substantially greater deflection occurs at the centre. This phenomenon also occurs in soil carrying uniformly distributed loading [4]; the usual model embodying a simple elastic foundation is quite unable to give an explanation for this phenomenon.

According to formulae (11) and (12) we have:

$$\frac{w\text{ (edge)}}{w\text{ (centre)}} = {}^1\!/_2(1 + e^{-sl})$$

The ratio of the above-mentioned deflections lies therefore between $^1/_2$ and 1.

Basic case c

Loading exerted by an infinitely rigid strip

The strip causes a uniform deflection w_0, with the result that a uniformly distributed reaction of magnitude kw_0 is developed. In addition, at the edges the resistance of the adjacent soil is transmitted as a concentrated line load.

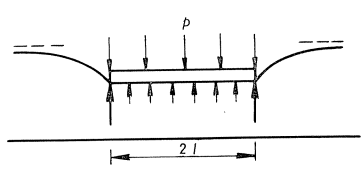

Fig. 7.

On the basis of the foregoing it is evident at once that this load is equal to kbw_0. The equilibrium of the strip then requires that:

$$p \cdot 2l = kw_0 \cdot 2l + 2kb\, w_0$$

so that:

$$w_0 = \frac{p}{k} \cdot \frac{l}{l+b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

Again b is found to act as the "co-operating width".

As experience shows, a concentration of bearing pressure occurs at the edges of rigid slabs supported on soil, although as a matter of course this concentration is levelled off by a local disturbance of the boundary equilibrium. The concentration effect is approximately represented by the line load at the edge.

1.3 Solutions for the axially symmetric case

As already noted in Section 1.1, the equation for the axially symmetric case can be solved with the aid of Bessel functions [5]. The functions and properties employed are summarised below. Next to the graphical representation, the first term (or terms) of an expansion in a series is given, both for small and for large values of x.

$K_0(x)$ and $I_0(x)$ are the functions primarily required; they are solutions of the equation:

$$w'' + \frac{1}{x} w' - w = 0$$

The other functions mentioned below occur on differentiating the K_0 and I_0 function or are useful for simplifying the boundary conditions.

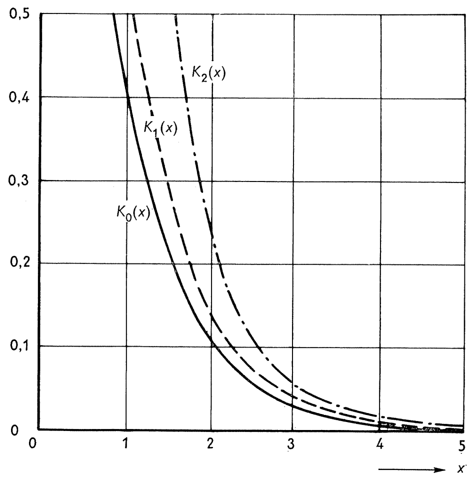

Fig. 8.

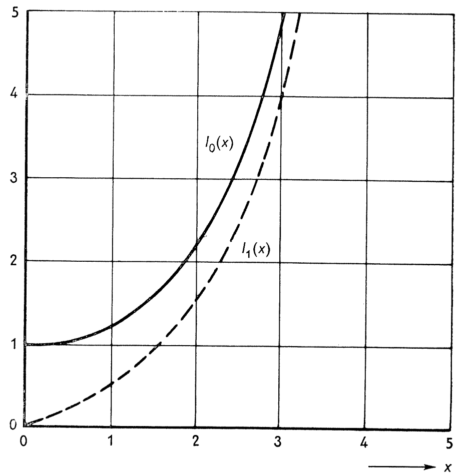

Fig. 9.

function	small x	large x
$K_0(x)$	$\ln \frac{2}{x}$	$\sqrt{\frac{\pi}{2x}}\, e^{-x}$
$K_1(x)$	$\frac{1}{x}$	$\sqrt{\frac{\pi}{2x}}\, e^{-x}$
$K_2(x)$	$\frac{2}{x^2}$	$\sqrt{\frac{\pi}{2x}}\, e^{-x}$

function	small x	large x
$I_0(x)$	$1 + \frac{1}{4}x^2$	$\frac{1}{\sqrt{2\pi x}}\, e^{x}$
$I_1(x)$	$\frac{1}{2}x$	$\frac{1}{\sqrt{2\pi x}}\, e^{x}$

The following differentiation rules are required:

$$I_0'(x) = I_1(x) \quad \text{(a)}$$

$$K_0'(x) = -K_1(x) \quad \text{(b)}$$

Furthermore, the following relations have been used for reducing and rearranging the boundary conditions:

$$I_0(x)K_1(x) + I_1(x)K_0(x) = \frac{1}{x} \quad \text{(c)}$$

$$xK_0(x) + 2K_1(x) = xK_2(x) \quad \text{(d)}$$

Basic case d

Loading by a point load

As appears from the graphs, the general solution for the axially symmetric case comprises a portion which decreases for increasing values of x (i.e., K_0) and a portion which tends to infinity for increasing values of x (i.e., I_0).

In the case of a concentrated load the deflection of the surrounding soil will diminish for increasing x, hence only the first solution remains:

$$w = W_1 K_0(sx)$$

The shear force on the wall of a cylinder with radius x is:

$$-2\pi x T = 2\pi A \cdot (-xw') = 2\pi A W_1 \cdot sx\, K_1(sx)$$

For small values of x this is approximately:

$$2\pi A W_1 \cdot sx \cdot \frac{1}{sx} = 2\pi A W_1, \text{ and hence is a constant.}$$

The inherent resistance of the spring itself under the load is negligible in comparison with the reaction caused by shear stresses, as it was in basic case a. The reaction for small values of x can therefore be equated to P. This gives the result:

$$W_1 = \frac{P}{2\pi A} = \frac{P}{2\pi k b^2}$$

The deflection is therefore known:

$$w = \frac{P}{2\pi k b^2} K_0(sx) \quad \dots\dots\dots\dots\dots \quad (14)$$

Here the co-operation of the adjacent soil cannot prevent the deflection under a theoretical point load from becoming infinitely large.

Basic case e

Distributed loading on a circular area with radius R

The solution for this case is obtained most rapidly by so determining the constants W_1 and W_2 that the continuity conditions for both w and w' at $x=R$ are satisfied. The solution consists of two branches, one for $x \leqslant R$ and one for $x \geqslant R$, as follows:

$$\left.\begin{aligned} &\text{if } x \leqslant R, \text{ then } w = \frac{p}{k}\{1 - sR \cdot K_1(sR) \cdot I_0(sx)\} \\ &\text{if } x \geqslant R, \text{ then } w = \frac{p}{k} \cdot sR \cdot I_1(sR) K_0(sx) \end{aligned}\right\} \quad \dots\dots \quad (15)$$

The continuity of w and w' is satisfied, since for $x = R$ we have:

$$w = \frac{p}{k}\left\{1 - sR\, K_1(sR)\, I_0(sR)\right\} = \frac{p}{k} \cdot sR \left\{\frac{1}{sR} - K_1(sR)\, I_0(sR)\right\} =$$

$$= \frac{p}{k} \cdot sR\, I_1(sR)\, K_0(sR), \text{ see equation (c) on page 36}$$

and:

$$w' = -\frac{p}{k} \cdot sR\, K_1(sR)\, I_0'(sR) = -\frac{p}{k} \cdot sR\, K_1(sR)\, I_1(sR) =$$

$$= \frac{p}{k} \cdot sR\, K_0'(sR)\, I_1(sR), \text{ see equations (a) and (b) on page 36}$$

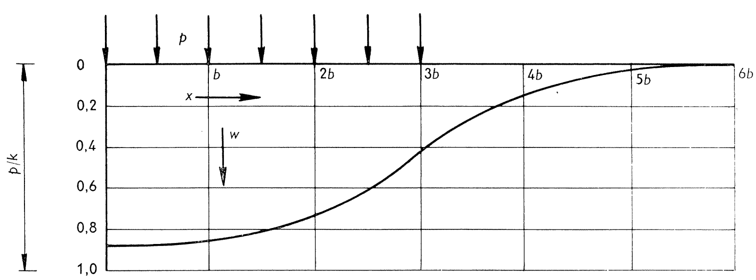

Fig. 10. Deflection curve for the surface of a foundation subjected to uniformly distributed loading p on a circular area with radius $3b$. The diagram is symmetrical with regard to the line $x = 0$.

The shape of the deflection curve depends on sR, i.e., on $R : b$.

The deflection curve for $R = 3b$ is shown in Fig. 10. Here, as in case b, the deflection at the centre of the loaded area is considerably greater than at the edge.

Basic case f

Loading exerted by an infinitely rigid circular slab

For the deflection of the unloaded area only the damping part of the solution can be used, so that there:

$$w = W_1 K_0(sx)$$

Hence if the slab is pressed downwards over a distance w_0:

$$W_1 K_0(sR) = w_0$$

At the edge of the slab a shear force is transmitted, whose magnitude per unit length along the circumference is:

$$-Aw' = AsW_1K_1(sR) = kb\ W_1K_1(sR)$$

The equilibrium of the slab is expressed by the following relation:

$$\pi R^2 p = \pi R^2 \cdot kw_0 + 2\pi R \cdot kb\ W_1K_1(sR)$$

or

$$p = kW_1 \left\{K_0(sR) + \frac{b}{R} \cdot 2K_1(sR)\right\}$$

Now according to equation (d) on page 36 we must have:

$$2K_1(sR) = sR\{K_2(sR) - K_0(sR)\}$$

On substituting this we obtain:

$$p = kW_1K_2(sR)$$

and finally:

$$w_0 = \frac{p}{k} \cdot \frac{K_0(sR)}{K_2(sR)} \qquad (16)$$

With a view to the practical application of this case the quantity:

$$f = \frac{K_0(sR)}{K_2(sR)} \quad \left(\text{factor in } w_0 = f\frac{p}{k}\right)$$

has been tabulated for a number of values of sR (page 39). The table need not be extended beyond $sR = 5$, because:

$$\text{for } R \gg b \text{ we have } w_0 = \frac{p}{k}\left(\frac{R}{R+b}\right)^2 \qquad (17)$$

Once more the significance of b as the co-operating width is to be noted.

Table 1. Plate bearing test ($x = sR = R : b$); values of f from formula $w = f\frac{p}{k}$

x	K_0	K_2	f	x	K_0	K_2	f
					$0.1\times$ ..	$0.1\times$ ..	
0.1	2.427	199.5	0.012	2.6	0.554	1.056	0.525
0.2	1.753	49.51	0.035	2.7	0.492	0.920	0.535
0.3	1.372	21.74	0.063	2.8	0.438	0.803	0.545
0.4	1.114	12.04	0.092	2.9	0.390	0.702	0.555
0.5	0.924	7.550	0.122	3.0	0.347	0.615	0.565
					$0.1\times$ ..	$0.1\times$ ..	
0.6	0.778	5.120	0.152	3.1	0.310	0.539	0.574
0.7	0.660	3.661	0.180	3.2	0.276	0.474	0.582
0.8	0.565	2.720	0.208	3.3	0.246	0.416	0.591
0.9	0.487	2.079	0.234	3.4	0.220	0.367	0.599
1.0	0.421	1.625	0.259	3.5	0.196	0.323	0.607
					$0.01\times$ ..	$0.01\times$ ..	
1.1	0.366	1.292	0.283	3.6	1.750	2.850	0.614
1.2	0.318	1.043	0.305	3.7	1.563	2.516	0.621
1.3	0.278	0.851	0.327	3.8	1.396	2.223	0 628
1.4	0.244	0.702	0.347	3.9	1.248	1.966	0.635
1.5	0.214	0.584	0.366	4.0	1.116	1.740	0.641
	$0.1\times$ ..				$0.01\times$ ..	$0.01\times$ ..	
1.6	1.880	0.489	0.384	4.1	0.998	1.541	0.648
1.7	1.655	0.412	0.402	4.2	0.893	1.366	0.654
1.8	1.459	0.349	0.418	4.3	0.799	1.211	0.660
1.9	1.288	0.297	0.434	4.4	0.715	1.075	0.665
2.0	1.139	0.254	0.449	4.5	0.640	0.954	0.671
	$0.1\times$ ..	$0.1\times$ ..			$0.01\times$ ..	$0.01\times$ ..	
2.1	1.008	2.177	0.463	4.6	0.573	0.848	0.676
2.2	0.893	1.874	0.476	4.7	0.513	0.754	0.681
2.3	0.791	1.617	0.489	4.8	0.460	0.670	0.686
2.4	0.702	1.400	0.501	4.9	0.412	0.596	0.691
2.5	0.623	1.215	0.513	5.0	0.369	0.531	0.695

Large x: $$K_0(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}\left(1 - \frac{1}{8x} + \frac{9}{128x^2} - \frac{75}{1024x^3}\cdots\right)$$

$$K_2(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}\left(1 + \frac{15}{8x} + \frac{105}{128x^2} - \frac{315}{1024x^3}\cdots\right)$$

$$f = 1 - \frac{2}{x} + \frac{3}{x^2} - \frac{15}{4x^3}\cdots$$

where $\left(\frac{x}{x+1}\right)^2 = 1 - \frac{2}{x} + \frac{3}{x^2} - \frac{4}{x^3}\cdots$ is a good approximation of f.

For example: if $x = 5$ then according to the table $f = 0.695$, while the approximation gives $f = (5/6)^2 = 0.694$.

2 Interpretation of the plate bearing test

2.1 General considerations

Rigid road and airfield pavements are analysed as slabs supported on spring-type elastic foundations. The modulus of subgrade reaction is determined by tests in situ. Various testing methods have been devised for the purpose [6]. Here only the plate bearing test will be considered. In this test a rigid circular plate is pressed down on the base under investigation, the force applied and the corresponding deflection being regularly observed. Plates of various diameters are used.

It has long been known that relatively higher values of the modulus of subgrade reaction are obtained with the smaller bearing plates. This is sometimes explained by considering the foundation not as a spring-type support but as an elastic medium. The plate bearing test can then be interpreted with reference to one of the known solutions provided by the theory of elasticity. The indentation of an isotropic elastic halfspace by a rigid circular plate is, for example, one of the cases for which a solution is known [7]. The solution for an anisotropic elastic halfspace has been given by KONING [8], and for an isotropic elastic layer on an infinitely rigid base the solution has been given by VOROVICH and USTINOV [9].

The use of one of these solutions in the interpretation of the plate bearing test has the disadvantage that subsequently the actual structure which is designed on the basis of the test result should also be analysed as supported on an elastic medium, with all the attendant complications.

For this reason a simpler method of interpretation is preferable. VREEDENBURGH proposed that a strip along the perimeter of the bearing plate be reckoned as belonging to the area of the plate. This is in agreement with the results of the theory of the coupled spring foundation for the basic cases a and c, and also for basic case f if sR is sufficiently large. Hence it appears that with these simple results the theory is in line with the intuitive visual approach with which an engineer likes to tackle his problems.

If it is additionally considered that several solutions of the differential equation of a beam or slab on a coupled spring foundation are already available, it will be evident that an interpretation of the plate bearing test on the basis of this theory is certainly worth considering.

2.2 The plate bearing test on a coupled spring foundation

The starting point for the following consideration of the problem is provided by the results for basic case *f*.

The test yields an apparent modulus of subgrade reaction k^*, which is obtained from the relation:

$$w = \frac{p}{k^*} \quad \text{and since } w = f\frac{p}{k} \quad \text{we have } \frac{k}{k^*} = f$$

The graph for $\frac{k}{k^*}$ can therefore be plotted from the data in Table 1.

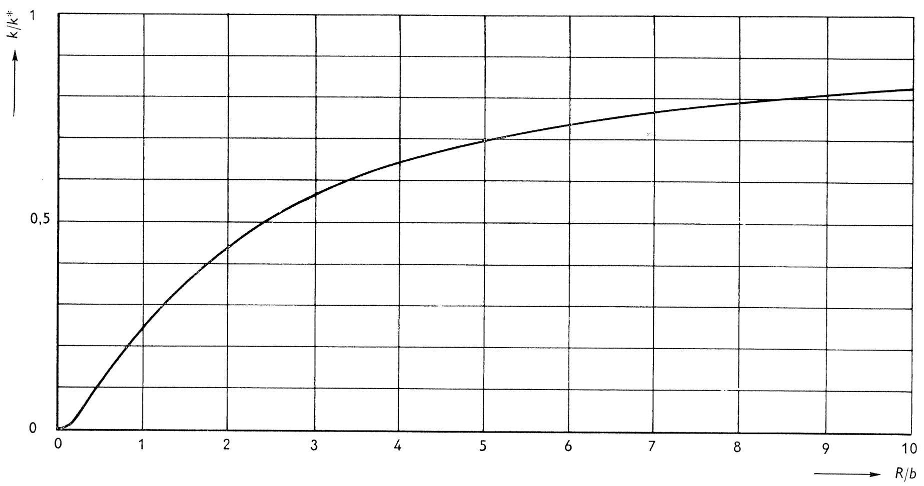

Fig. 11. The ratio between the true modulus of subgrade reaction k and the apparent modulus of subgrade reaction k^* plotted as a function of the radius R (of the bearing plate by means of which k^* was determined) divided by the co-operating witdh b.

Now k could be determined from k^* if the ratio R/b were known. However, the constant b is also dependent on the properties of the subgrade and must therefore likewise be determined from measurements.

If test results for plates with various diameters are available, a curve of the shape shown here can be made to conform as closely as possible to the measured values by trying a number of different values of b. A difficulty lies in the fact that the plates used in the bearing test are not infinitely rigid. If the deflection obtained for the centre of the plate is different from the deflection at the edge, it will still be necessary to apply a certain conversion based on splitting up the total bearing resistance of the soil into bearing pressure under the plate and a reaction along the edge:

$$P = Fp = Fk^*w = Fk\frac{1}{f}\,w = Fkw + Fk\frac{1-f}{f}\,w$$

In the first term on the right-hand side we must now substitute the estimated average of the settlement under the plate, w_p, and in the second term the settlement w_r of the edge.

In general the bearing reaction of a plate is found to be:

$$P = Fkw_p + Fk\frac{1-f}{f}\,w_r$$

In order to simplify the elaboration, an equivalent settlement w_g is defined, this being the value that would be obtained in a test with an infinitely rigid plate having the same dimensions and subjected to the same load P. It is equal to:

$$w_g = f w_p + (1 - f) w_r, \text{ for then } Fk \frac{1}{f} w_g = P.$$

By trial and error with different values of b we progressively obtain better values for f.

In the analysis the scatter of the test results presents some difficulties. A probably more accurate method of determining a value for b consists in measuring the deflection curve of the surface adjacent to the plate.

According to the results for basic case f this deflection curve is given by:

$$w = \frac{p}{k} \frac{K_0(sx)}{K_2(sR)} = w_0 \frac{K_0(sx)}{K_0(sR)}$$

The resulting deflection curves, shown in Figure 12, are found to differ sufficiently for different values of b, so that a dependable estimate of b can be obtained from the results of the measurements.

At all events, it must be borne in mind that a stress concentration occurs at the edge and that this may disturb the boundary equilibrium, with the result that the material properties are altered and therefore k and b also. For this reason the deflection curve should be measured at the smallest possible amount of settlement.

As already stated in the Introduction, the model investigation of airfield runways gave occasion to consider the co-operating width effect more closely.

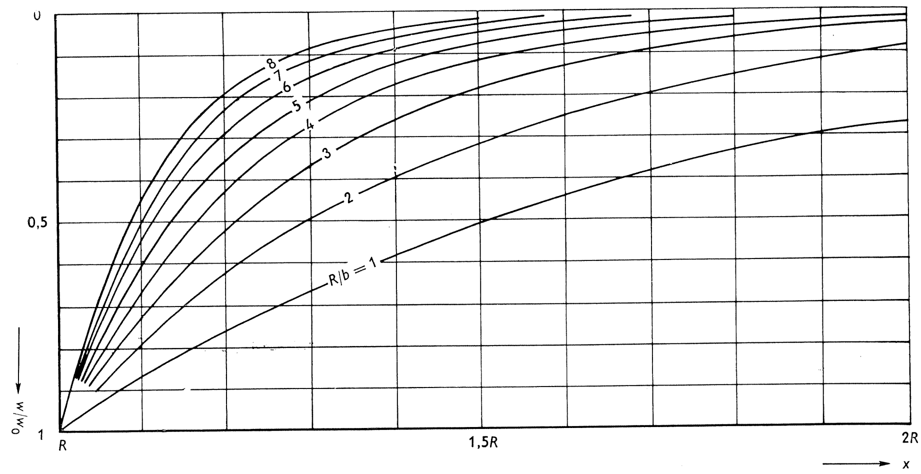

Fig. 12. Curves representing the deflection adjacent to the bearing plate (expressed as a fraction of the settlement of the plate itself) for various ratios of the radius R to the co-operating width b.

If the co-operating widths of the actual subgrade and of the model material which simulates the soil are known, the model scale can be deduced. The question now is whether the behaviour of the soil and of the model material can be described in terms of a coupled spring foundation.

2.3 Results for soil

The following is based on information supplied by Ir. H. van der Most and Ir. A. Jonker, formerly associated with the Soil Mechanics Laboratory at Delft. Extensive scanning of the literature regarding k^*-values measured with bearing plates of various sizes produced the data which are summarised in Fig. 13. This diagram corresponds essentially to Fig. 11, but as k and b are not known (and will moreover vary for the different tests), k/k^* and R/b cannot be adopted as co-ordinates.

For the vertical axis the apparent modulus of subgrade reaction k^* has, just as in Fig. 11, been placed in the denominator, but the numerator now contains the value of k^* for the plate of 75 cm (30″) diameter. The result is that the curves must now pass through the point (75; 1.0), but different curves are still possible for different values of b.

By way of general indication, the theoretical curves for $b = 5$ cm and $b = 40$ cm are presented. On comparing these with the test results one sees that in the range of small plate diameters all the tests do in fact show an upward trend, but the possible approach to a horizontal asymptote for larger plate diameters is by no means clear.

The results for different soils vary quite considerably, which makes the comparison rather difficult.

Good agreement with the theory described here is shown by the results of tests performed at a single site at the request of the A.S.C.E. [10]. These are embodied in Fig. 14.

It can be inferred that for loaded areas of the size such as are considered in the design calculations of airfield runways, the theory of the coupled spring foundation gives a reasonably good picture of the variation of the resistance with the magnitude of the loaded area. The more critical question as to whether it is possible to use the values of k and b determined with small plates as a basis for extrapolation to very large loaded areas can be answered only when more extensive research has been carried out. Also, it would be desirable to incorporate the non-elastic properties into a concept of the supporting behaviour of soils.

2.4 Results for a model material

A large number of model materials were investigated for the purpose of the model tests in connection with the researches reported in [11]. Here only the results for a strong foam rubber with open pores will be given (this is not the

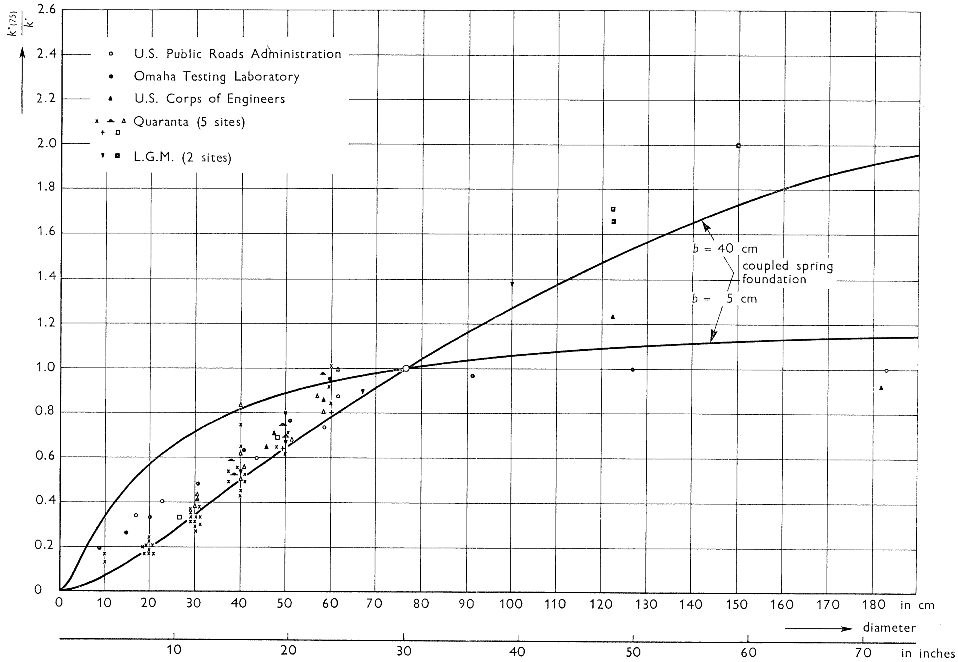

Fig. 13. Experimental results of plate bearing tests on soil.

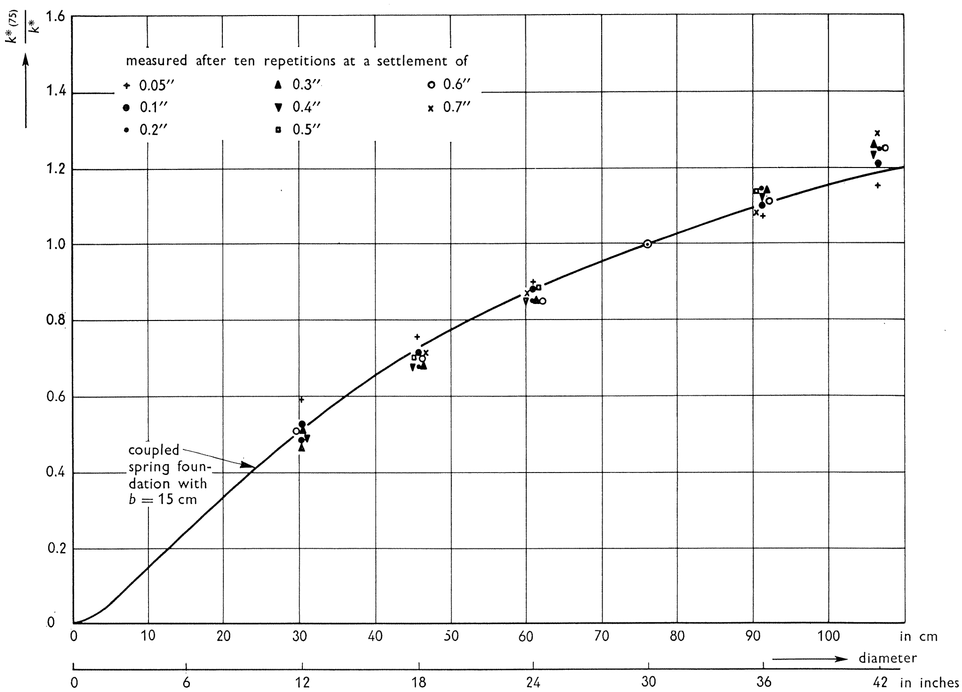

Fig. 14. Comparison of the experimental results of the A.S.C.E. tests with the results yielded by the theory of the coupled spring foundation.

material that was finally selected for the airfield runway model; it was, however, the most suitable material for accurate tests relating to the co-operating width effect).

For a model material it is a much simpler matter to determine the true value of k than it is for soil. In the former case it is merely necessary to load a small specimen of the material with a bearing plate of the same size. The value measured for a specimen measuring 3.4 cm $\times$ 10 cm, with a thickness of 1.8 cm, was $k = 2.12$ kg/cm^3.

A plate bearing test with a 25 cm^2 plate was carried out on a specimen measuring 50 cm $\times$ 50 cm (therefore to be regarded as very large with respect to the plate). The deflection curve adjacent to the plate was determined by means of a photographic measuring technique developed by Mr. P. G. JEU-

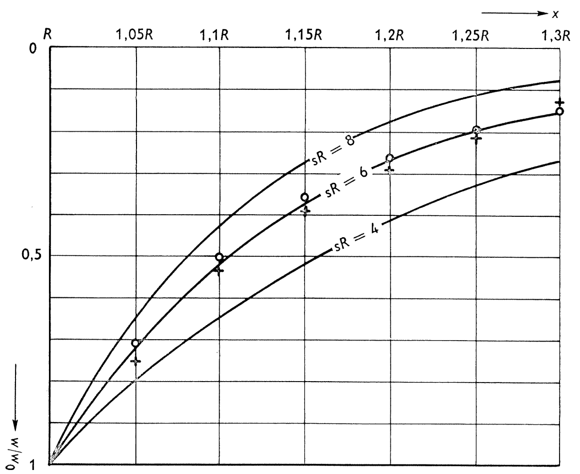

Fig. 15. Comparison of the deflection curve adjacent to a bearing plate on model material (foam rubber) with the theoretical deflection curve for a coupled spring foundation. (Measured from photograph: + left side, o right side).

NINK of the Stevin Laboratory. The ratio w/w_0 has been plotted in Fig. 15. The curve thus obtained is found to be in very good agreement with the theoretical line for $sR = 6$. Hence it follows that $b = 0.47$ cm.

For $sR = 6$ we have $f = (6/7)^2 = 0.735$. Therefore k^* would have to be: $k^* = k : f = 2.88$ kg/cm^3. The measured value is $k^* = 3.06$ kg/cm^3, i.e., only 6 per cent greater. So for this material the theory of the coupled spring support is found to be in excellent agreement with the actual conditions.

2.5 *The conversion of results obtained from model tests*

When the parameters k and b for soil and for the model material have been determined by means of field tests, on the one hand, and by laboratory tests, on the other, we are faced with the problem of designing the model with due regard to the laws of similarity between the model and its prototype.

Starting from the differential equation of the rigid slab on a coupled spring foundation, a number of characteristic values (nondimensional numbers N) can be established which must have the same value for the model and for the prototype.

$$D\Delta\Delta w + kw - A\Delta w = p$$

$$N_1 = l_c/L; \; N_2 = b/L; \; N_3 = p/kw \qquad (18)$$

where: $D = Et^3/12(1-\nu^2)$ bending stiffness of the slab

$l_c = \sqrt[4]{4D/k}$ characteristic length

L an arbitrary dimension of the slab surface

The characteristic value N_2 must first be considered. It here appears that the co-operating width of the model material determines the linear scale of the model. However, it may well occur that, for other reasons, this linear scale is not convenient for the investigation. It is therefore necessary to have available a large number of model materials with various values for the co-operating width, in order to offer a choice with regard to the linear scale.

The characteristic value N_1 can be made equal for the model and for the prototype without difficulty, since a separate scale factor can be chosen for the thickness of the slab, independently of the linear scale of the dimensions of the slab surface. Finally, the characteristic value N_3 is of no direct significance with regard to the design of the model; it arises only in connection with the elaboration of the test results.

If visco-elasticity or non-linear phenomena have to be taken into account in the model analysis, additional characteristic values play a part. In general, however, these complications are ignored or they are taken into account by indirect means.

An example is provided by the model tests which were carried out in connection with the investigation of runway pavements at Schiphol Airport, Amsterdam. In that case the model analysis preceded extensive measurements performed on the actual structure. The results obtained with the model were used for determining the correct location of the measuring devices and the route to be travelled by the test-loading vehicle.

To enable this to be done, influence surfaces for the bending moment at a large number of points of the model were determined. The use of the concept of influence surfaces for a unit load (whence the results for various loading groups were deduced) in itself already implies the assumption that the law of superposition holds and that non-linear effects can be left out of account. This assumption was based on the consideration that the soil stresses will remain small in the actual test. Also, it was considered that the load duration would be small, so a time effect can hardly be expected.

Fig. 16. Test on the airfield runway model.

A number of plate bearing tests in situ with plates of various diameters enabled the deduction of the coefficient of subgrade reaction and the co-operating width. The readings were taken at small values of the settlement, in order to obviate non-linear phenomena, and for a short load duration (corresponding to the load duration with normal traffic on the runway), in order to eliminate the time effect. The tests on the model material were performed in similar fashion. The model law for the linear scale in this case led to the choice of a supporting material (sponge rubber) with a rather considerable time effect; a better material was available but would have required a model of impracticable size. To overcome the difficulties associated with the time effect, it was necessary to employ a special measuring procedure which, together with the measuring instrument used, was developed by Ir. H. M. DE HAAS of the Stevin Laboratory. Results of the experiment, comparison with other investigations, and further literature references are given in the publication of JONKER and VAN NIEUWENHUYZEN [11].

3 Other possible applications

A mathematical model should not be extended unless there are compelling reasons for doing so. In all cases where good structural design criteria can be obtained with an analysis assuming an ordinary spring-type elastic foundation, it is superfluous to take the coupling effect into account. All the same, it may still be worth while to remember this effect exists. In the analysis of, for example, a beam resting on soil it is possible to take account of the fact that on each side of the beam (i.e., in the direction perpendicular to the axis) the material over a certain distance, equal to the co-operating width, helps to develop the bearing reaction. Especially in the case of a narrow beam, the bearing reaction will be larger and the stress distribution probably more favorable than expected.

Other examples can be imagined, in which the coupling effect has unfavourable consequences. Then safe design requires taking it into account. Take, for instance, a beam with one free end and the other end built in. When the built in end undergoes a settlement, bending moments will develop, whose magnitude depends on the distribution of the counterpressure beneath the beam. The coupling effect will cause part of the reaction to be concentrated near the free end, so the bending moments attain larger values than would otherwise be the case.

The analysis of a structure supporting a uniformly distributed load, such as the bottom of a liquid storage tank, also calls for some attention. On an ordinary elastic foundation the bottom would undergo uniform displacement, so that no bending moments would be produced. But as a result of the coupling in the foundation the settlement becomes non-uniform, as in basic case e. Since bending moments will then indeed occur, it is certainly advisable to take the coupling effect into consideration in this case.

Highway engineers are, as a rule, well aware of the coupling effect, which is reflected in the analysis of highway pavements and their foundations as elastic multilayer systems. A particularly complex mathematical model has been chosen for the purpose. The present author is unable to judge whether this is indeed necessary. It is, however probably useful to examine the relation between these systems and the coupled spring foundation. By way of example a single elastic layer (of thickness h) on a completely rigid base will be considered.

If the entire site is covered by uniformly distributed loading, then:

$$w = \frac{ph}{E}\frac{(1-2\nu)(1+\nu)}{1-\nu}, \text{ so that } k = \frac{E}{h}\frac{1-\nu}{(1+\nu)(1-2\nu)} \quad \ldots\ldots \quad (19)$$

A similar result can be derived for other multilayer systems. Next, the deflection of very small loaded areas is considered. Pressure applied to a circular rigid plate produces a deflection which, for a very small plate, is determined by the stresses in the top part of the upper layer and is therefore equal to the deflection in the case of an elastic halfspace:

$$w = \frac{pR}{E} \cdot {}^1/_2\pi(1-\nu^2)$$

Now the deflection of a similar small plate on a coupled spring foundation is likewise approximately proportional to the radius R, as is apparent from Fig. 11. The following approximation is valid:

$$\text{for} \qquad 0 < R < 1.7b \ \text{ we have } \ w = 0.245 \frac{pR}{kb} \quad \ldots\ldots\ldots\ldots \quad (20)$$

On comparison with the result for the elastic layer it appears that we can write:

$$b = 0.156\, h \frac{1-2\nu}{(1-\nu)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

For example, for $\nu = {}^1/_3$ we have: $k = 4.5E/h$ and $b = 0.117h$. The result obtained for loading on a coupled spring foundation with these parameters is approximately the same as that for loading on the elastic layer.

Because of the simplification of the mathematical model it can now be considered whether other effects can also be taken into account, such as visco-elasticity of the subgrade or of the structure itself.

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