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THE ANALYSIS OF PLATES OF ABRUPTLY VARYING THICKNESS WITH THE AID OF THE METHOD OF DIFFERENCES

The well-known method of differences for the analysis of plates of constant thickness, which until a few years ago was very laborious, deserves renewed attention because the advent of the electronic computer has removed this drawback.
It is now also of interest to extend the method to plates whose thickness changes abruptly at regular distances in two directions. Plates with apertures are a special case. It is shown that the difference equation for a plate of abruptly varying thickness, as derived in this paper, can also be used for establishing the conditions for arbitrary boundaries. This equation gives directly, i.e., without having to introduce "external" points as is usually done in the existing literature, a difference equation for the points at the edge and the points located at a distance of one mesh width therefrom.
The theory is further elucidated with the aid of a worked example, and the results are verified by means of tests performed on a model by employing the moiré method.
Attention is called to the possibility of programming the entire procedure.

0 Introduction

In isotropic plates of constant thickness, and therefore of constant stiffness with respect to bending, the state of stress under load is governed by the well-known biharmonic differential equation of the displacements w perpendicular to the plane of the plate:

$$K\Delta\Delta w = q \qquad (1)$$

where q is the load, K is the flexural stiffness of the plate, and Δ is the Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad K = \frac{Eh^3}{12(1-\nu^2)} \qquad (2)$$

Furthermore, h denotes the thickness of the plate, E the modulus of elasticity of the material, and ν is Poisson's ratio. Analytical solutions of the differential equation (1) are known for suitably chosen plate shapes and boundary conditions usually not of a complicated character (see, inter alia, the extensive bibliography in [1] and [2].

In those cases where the plate has a more complicated perimeter and the boundary conditions along the perimeter can vary as well, it is usually not possible to arrive at a straightforward analytical solution by a simple procedure, and often such a solution is not possible at all.

One of the methods to which one can then have recourse is the calculus of differences. In that case the plate is not treated as a continuum. The analysis is, instead, confined to a number of suitably chosen points of the plate whose displacements w are introduced as unknowns [1, 2, 3]. At each point the differential equation [1] is replaced by a linear equation in the displacements of that point and of a number of adjacent points.

If this procedure is performed for n points of the plate, the problem of solving the differential equation (1) is reduced to solving n independent linear equations with n unknowns.

The method is attractive only if such a set of n equations can be solved in a simple manner.

Before the development of electronic computers a direct solution could be obtained in something like a reasonable time only for $n \leqslant 20$, so that efforts were made to find possibilities for solving the sets of equations by iterative methods. SOUTHWELL's relaxation method [4] can be mentioned as an example of this.

Since these procedures were still very laborious, the analysis of plates with the aid of the calculus of differences was, until fairly recently applied only to a limited extent. However, now that electronic computers are available, this difficulty has been removed, and it has become an attractive proposition to make more ample use of this method.

For this reason, too, it is advantageous to extend the method to plates in which abrupt changes of thickness occur. The need for this was felt in analysing the stress distribution in a strip floor supported on columns [5] and floors with cantilevered balconies [6].

1 Application of the calculus of differences to plates of constant thickness

The procedure for the calculus of differences as applied to the determination of the action of forces in plates of constant thickness is, briefly, based on the following:

For plates of constant thickness the differential equation is:

$$K\Delta\Delta w = q \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (1)$$

The expressions for the moments (m) and shear forces (q) are:

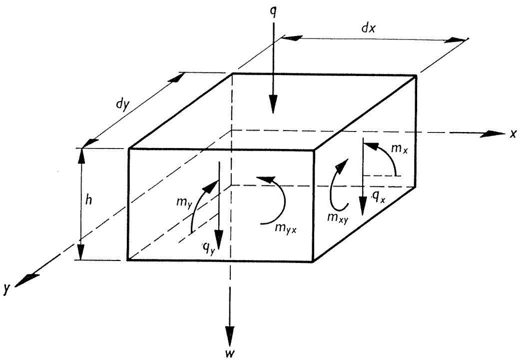


Fig. 1.

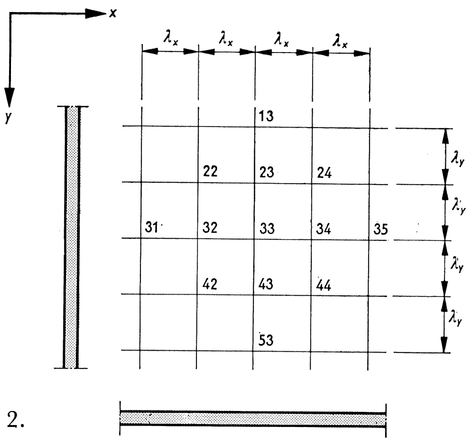


Fig. 2.

$$m_x = -K\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right] \qquad q_x = -K\frac{\partial}{\partial x}\Delta w$$

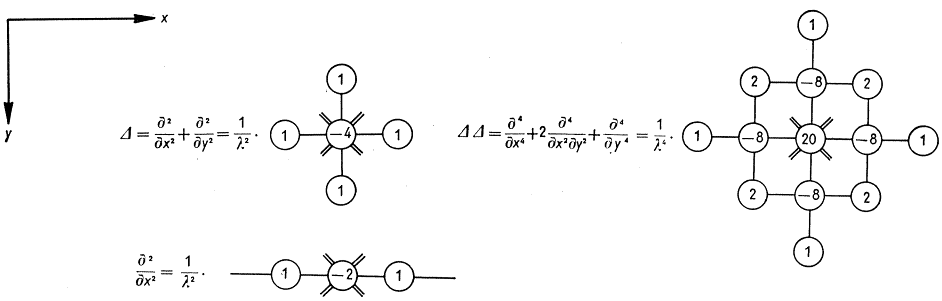
$$m_y = -K\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \qquad q_y = -K\frac{\partial}{\partial y}\Delta w \qquad \ldots\ldots\ldots \quad (3)$$

$$m_{xy} = -K(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

The sign convention for these quantities is defined in Fig. 1.

In the method of differences the plate is disposed in a difference network, usually a pattern of rectangles or squares formed by straight lines (see Fig. 2). The intersection of two net lines is called a 'net point'. The single plate portion enclosed by four lines is referred to as a 'panel'. The difference equation for a net point can be directly derived from the differential equation (1) by transforming the differential operations into difference operations.

For a square network ($\lambda_x = \lambda_y = \lambda$) the principal arrays of differences, referred to the co-ordinate axes of Fig. 2, are indicated in Fig. 3 (see, for example, ref. [2]).



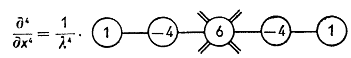


Fig. 3. Central difference arrays in replacement of differential expressions at the central point.

Thus, for net point 33 the difference equation (array ΔΔ) is as follows:

$$K[20w_{33}-8(w_{23}+w_{32}+w_{43}+w_{34})+2(w_{22}+w_{42}+w_{44}+w_{24})+ \\ +(w_{13}+w_{31}+w_{53}+w_{35})] = q\lambda^4 \quad \ldots . \quad (4)$$

For plates with abrupt changes in thickness it is not possible to describe the behaviour by means of one differential equation. Hence in that case the difference equation cannot be derived from a differential equation. The equation in question can, however, be directly derived, as will be done in Chapter 2.

In order to enable the reader to familiarise himself with the procedure that will then be employed, this derivation will now first be given for flat plates. To this end, an analogy model will be used whose mode of functioning corresponds to that of the plate schematised according to the calculus of differences (Fig. 4), lateral contraction being neglected. The load, which is conceived as being concentrated at a number of regularly spaced points of the plate, is transmitted by both bending and torsion.

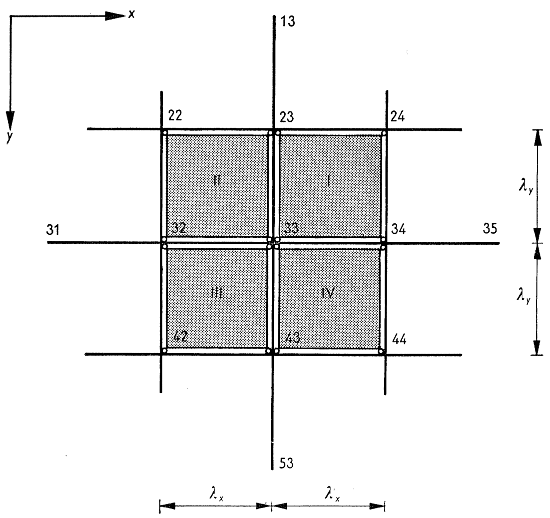


Fig. 4. Analogy model, comprising flexurally stiff beams and torsional panels, for a flat plate loaded in bending and torsion.

In the model the plate is replaced by a number of flexurally stiff beams of zero torsional stiffness which coincide with the lines of the difference network. The flexural stiffness of these beams is λK. In the model the torsional share of the load-transmitting action is provided by plates with stiffness K which are attached by means of hinged connections to the intersection points of the beam grillage formed in this way[1]). Consider the equilibrium of point 33 (Fig. 4).

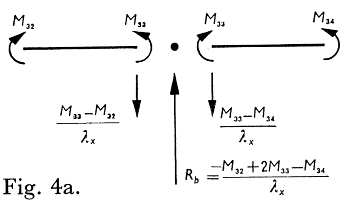


Fig. 4a.

The load at point 33 is $q\lambda_x\lambda_y$, where q is the load per unit area and λ_x and λ_y are the mesh width in the x-direction and y-direction respectively.

[1]) A model of this kind is employed by LIGHTFOOT for the analysis of plates by means of the grid framework method. However, he considers the model to be unsuitable in this form, and he distributes the torsional stiffness of the panels over the adjacent beams and uses the resulting grid framework for his calculations (LIGHTFOOT, E.: A grid framework analogy for laterally loaded plates. Int. J. Mech. Sci., 1964, Vol. 6, pp. 201–208).

This load must be in equilibrium with the sum of the shear forces transmitted by the beams (R_b) and by the reactions that the torsional panels produce at point 33 (R_w), so that:

$$q\lambda_x\lambda_y = R_b + R_w \quad \ldots \quad (5)$$

The sum of the shear forces transmitted by the beams (R_b) is (Fig. 4a):

$$R_b = \frac{-[M_x]_{32}+2[M_x]_{33}-[M_x]_{34}}{\lambda_x} + \frac{-[M_y]_{43}+2[M_y]_{33}-[M_y]_{23}}{\lambda_y} \quad \ldots \quad (6)$$

Next, the moments of equation (6) will be expressed in the displacements of the net points.

The moment in the x-direction at point 33 is:

$$[M_x]_{33} = -\lambda_y K \varkappa_x, \quad \text{thus} \quad \varkappa_x = -\frac{[M_x]_{33}}{\lambda_y}\frac{1}{K}$$

The relation between the moment and the displacements will now be established by introducing the approximation that the curvature $\varkappa_x$ has a constant value over the adjacent beam portions 32–33 and 33–34 respectively. The moment will then be proportional to the rise of the arc 32–33–34 of the deflection curve, which is equal to:

$$w_{33} - \tfrac{1}{2}(w_{32}+w_{34}) = -\tfrac{1}{2}(w_{32}-2w_{33}+w_{34})$$

Suppose point 33 is horizontally restrained; then:

$$w_{32} = \tfrac{1}{2}\varkappa_x\lambda_x^2; \quad w_{33} = 0; \quad w_{34} = \tfrac{1}{2}\varkappa_x\lambda_x^2$$

Therefore:

$$\frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2} = \varkappa_x$$

Expressed in the moment at point 33:

$$\frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2} = -\frac{[M_x]_{33}}{\lambda_y}\frac{1}{K}$$

Hence:

$$[M_x]_{33} = -\lambda_y K \frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2}$$

The moments occurring in equation (6) are then:

$$\left.\begin{aligned}
[M_x]_{32} &= -\lambda_y K\,\frac{w_{31}-2w_{32}+w_{33}}{\lambda_x{}^2}\\
[M_x]_{33} &= -\lambda_y K\,\frac{w_{32}-2w_{33}+w_{34}}{\lambda_x{}^2}\\
[M_x]_{34} &= -\lambda_y K\,\frac{w_{33}-2w_{34}+w_{35}}{\lambda_x{}^2}\\
[M_y]_{43} &= -\lambda_x K\,\frac{w_{53}-2w_{43}+w_{33}}{\lambda_y{}^2}\\
[M_y]_{33} &= -\lambda_x K\,\frac{w_{43}-2w_{33}+w_{23}}{\lambda_y{}^2}\\
[M_y]_{23} &= -\lambda_x K\,\frac{w_{33}-2w_{23}+w_{13}}{\lambda_y{}^2}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (7)$$

Substitution of equation (7) into equation (6) gives:

$$R_b = \lambda_y K\,\frac{w_{31}-4w_{32}+6w_{33}-4w_{34}+w_{35}}{\lambda_x{}^3} + \lambda_x K\,\frac{w_{53}-4w_{43}+6w_{33}-4w_{23}+w_{13}}{\lambda_y{}^3} \quad (8)$$

For determining the reactions produced by the torsional panels at the net points consider Fig. 4b. The hinged connections transmit vertical forces (P) which, for reasons of equilibrium, can have no other distribution than that indicated in Fig. 4b. The panel is therefore subjected to NADAI's loading case, in which a twist $\varkappa_{xy}$ constant over the area of the panel and equal to the following value (e.g., for panel I) occurs:

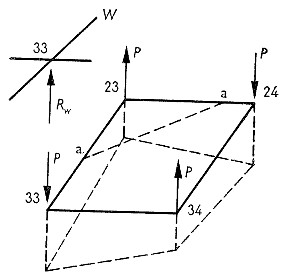


Fig. 4b.

$$[\varkappa_{xy}]_{\mathrm{I}} = \frac{-w_{33}+w_{34}-w_{24}+w_{23}}{\lambda_x\lambda_y}$$

The torsional moment in this panel is then:

$$[m_{xy}]_{\mathrm{I}} = -K\,\frac{-w_{33}+w_{34}-w_{24}+w_{23}}{\lambda_x\lambda_y} \quad \ldots\ldots\ldots\ldots \quad (9)$$

A section at an angle of 45° with respect to the edges of the panel is subjected to a bending moment having the value $\frac{1}{2}P$. The moment acting upon a section perpendicular thereto is of the same magnitude but with the opposite algebraic sign. The moments under consideration are the principal moments in the plate; the sum of the principal moments is therefore zero. The distributed shear force is proportional to the first derivative of the sum of the principal moments and is therefore also zero. From the equilibrium of a portion of the panel cut off by the line *a—a* (Fig. 4b) it is, however, apparent that a total shear force of magnitude P has to be transmitted. Since the distributed shear force is zero, a concentrated shear force of magnitude $\frac{1}{2}P$ must act upon the edges. A section parallel to one of the sides of the panel is therefore subjected to a distributed moment m_{xy} as also two opposite concentrated shear forces $\frac{1}{2}P$. From the equilibrium of the portion cut off by the section it then follows that:

$$m_{xy}\lambda + \tfrac{1}{2}P\lambda = P\lambda$$

so that:

$$P = 2m_{xy}$$

or more particularly for panel I:

$$P_{\mathrm{I}} = 2[m_{xy}]_{\mathrm{I}} = 2K\,\frac{w_{33}-w_{34}+w_{24}-w_{23}}{\lambda_x\lambda_y} \quad \ldots\ldots\ldots\ldots (10)$$

The total reaction produced at point 33 is:

$$\begin{aligned} R_w &= P_{\mathrm{I}}-P_{\mathrm{II}}+P_{\mathrm{III}}-P_{\mathrm{IV}} = \\ &= 2K\left[\frac{w_{33}-w_{34}+w_{24}-w_{23}}{\lambda_x\lambda_y}+\frac{w_{33}-w_{23}+w_{22}-w_{32}}{\lambda_x\lambda_y}+\right. \\ &\qquad \left.+\frac{w_{33}-w_{32}+w_{42}-w_{43}}{\lambda_x\lambda_y}+\frac{w_{33}-w_{43}+w_{44}-w_{34}}{\lambda_x\lambda_y}\right] \\ &= 2K\,\frac{4w_{33}-2w_{34}-2w_{23}-2w_{32}-2w_{43}+w_{24}+w_{22}+w_{42}+w_{44}}{\lambda_x\lambda_y} \quad \ldots (11) \end{aligned}$$

Substitution of equations (8) and (11) into equation (5) yields the difference equation for point 33:

$$\begin{aligned} &K\left[\frac{w_{31}-4w_{32}+6w_{33}-4w_{34}+w_{35}}{\lambda_x^4}+\frac{w_{53}-4w_{43}+6w_{33}-4w_{23}+w_{13}}{\lambda_y^4}+\right. \\ &\left.+\frac{8w_{33}-4w_{34}-4w_{23}-4w_{32}-4w_{43}+2w_{24}+2w_{22}+2w_{42}+2w_{44}}{\lambda_x^2\lambda_y^2}\right] = q \quad \ldots (12) \end{aligned}$$

For a square difference network ($\lambda_x = \lambda_y = \lambda$) the equation (12) is identical with equation (4).

The equation is valid for all net points that are located at a distance of at least two mesh widths from the edge. For points at one mesh width from the edge and for points at the edge the difference equation must be modified because these points undergo the effect of the edge. How this can be done will be dealt with in detail later on.

The difference equations for all the net points together form a set of linear equations in the displacements. An electronic computer will mostly be used for solving this set.

When the displacements have thus been obtained, the moments and shear forces can be determined from them.

The bending moments in the beams of the analogy model can be determined with the aid of equation (7). These moments can be expressed in terms of plate moments per unit width.

Thus, at point 33 of Fig. 2 the following expressions are obtained for the bending moments:

$$\left.\begin{aligned} [m_x]_{33} &= -K\frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2} \\ [m_y]_{33} &= -K\frac{w_{23}-2w_{33}+w_{43}}{\lambda_y^2} \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \ldots \quad (13)$$

The constant torsinal moments in the panels of the analogy model can be determined with the aid of equation (9) and be directly used for the plate. For example, for panel 33–32–22–23:

$$[m_{xy}]_{\text{I}} = -K\frac{w_{33}-w_{32}+w_{22}-w_{23}}{\lambda_x\lambda_y} \quad \ldots \ldots \ldots \ldots \quad (14)$$

For determining the shear forces in the analogy model, the share thereof due to torsion will first be examined more closely. The reactions P at the corners of the torsional panels cause shear forces of magnitude $\frac{1}{2}P$ which – as has been shown in the foregoing – manifest themselves as constant concentrated forces along the edges. In 'translating' the shear forces in the analogy model into those for the actual plate, it must be borne in mind that these concentrated shear forces partly cancel one another along the panel sides which they have in common.

Thus if, for example, the panels II and III undergo the same amount of tilt, the shear forces $-\frac{1}{2}P_{\text{II}}$ and $\frac{1}{2}P_{\text{III}}$ will just cancel each other. With different amounts of tilt of the panels, as will generally be the case, the resultant shear force in the x-direction will be:

$$Q_x = \tfrac{1}{2}P_{\text{III}} - \tfrac{1}{2}P_{\text{II}}$$

Since no discontinuity can be expected to occur at the side shared in common by the panels, this shear force, converted for the plate, is uniformly distributed, so that:

$$q_x = \frac{1}{2}\,\frac{P_{\text{III}}-P_{\text{II}}}{\lambda_y}$$

The shear forces in the beams of the analogy model, converted for the plate, are likewise uniformly distributed. The sum of the shear forces in the analogy model, e.g., in the portion 32–33, in the beam and the adjacent torsional panels II and III is:

$$Q_x = \frac{[M_x]_{33}-[M_x]_{32}}{\lambda_x} + \tfrac{1}{2}P_{\text{III}} - \tfrac{1}{2}P_{\text{II}}$$

For the plate, expressed in shear force per unit width:

$$q_x = \frac{[m_x]_{33} - [m_x]_{32}}{\lambda_x} + \frac{[m_{xy}]_{III} - [m_{xy}]_{II}}{\lambda_y} \quad \ldots\ldots\ldots\ldots (15)\ ^{1)}$$

2 Plates with abrupt changes of thickness

Fig. 5 shows part of a plate with abrupt, i.e., stepwise, changes of thickness; the usual difference network is also shown. At the difference lines the thickness of the plate may undergo a sudden increase or decrease. In the most general case all the panels will differ from one another in respect of thickness. Each panel in the diagram is indicated by a Roman numeral. The stiffness of a panel is denoted by K followed by the relevant Roman numeral as a subscript.

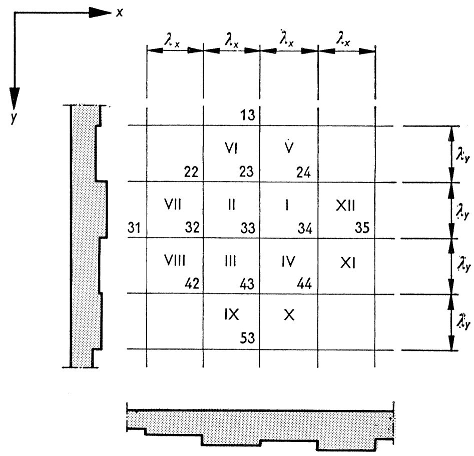


Fig. 5.

In the derivation of the difference equation the effect of the flange action and of lateral contraction (Poisson's ratio) will be ignored.

With regard to flange action the following can be noted. The starting point in the derivation of the elementary plate equation is that the loading gives rise only to moments and shear forces. In slabs whose middle surface is not a flat plane – e.g., because the edges are thicker than the central portion – 'flange action', however, additionally produces normal (direct) forces in the plane of the plate. Although it is possible in principle to introduce flange action into an analysis according to the method of differences, an investigation of the problem [7] revealed that in most cases this action can quite safely be left out of account.

The analogy model already described will again be used for establishing the difference equation (Fig. 5). The difference in relation to the flat plate (of constant thickness), however, is that now the flexurally stiff beams as well as the torsional panels all have different stiffness values. Since the beams of the model are assumed to have zero torsional stiffness, they can be conceived as being composed of two beams disposed freely side by side and each having a stiffness equal to $\frac{1}{2}\lambda$ times the plate stiffness of the adjacent parts of the plate.

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[1]) By expressing the moments of equation (15) in the displacements and then rearranging them, the well known formula is obtained:

$$q_x = \frac{[m_x + m_y]_{33} - [m_x + m_y]_{32}}{\lambda_x}$$

At the net points the beams which are *in a direct line with each other* are assumed to be joined to each other by a flexurally stiff connection. In this way nodes (junctions) of four beams which cross one another at right angles and are all separate from one another, but which undergo the same displacement, are obtained. From the following treatment of the problem it will appear why the beams in the analogy model are interconnected in this particular way.

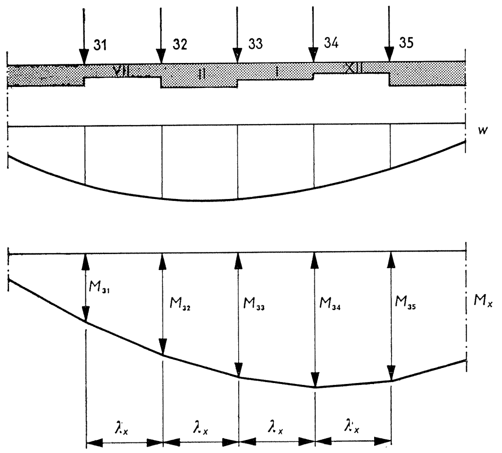


Fig.6.

As the thickness of the plate does not vary within a mesh of the difference network, the stiffness of each torsional panel is constant. On considering the equilibrium of point 33 it is apparent that the same equations as those which were derived for the flat plate (equations (5) and (6)) are valid for this case, too, since equilibrium equations are not affected by variations of thickness. The next step consists in expressing the moments of equation (6) in terms of displacements of the plate.

Consider the bending, in the x-direction, of the beam located on the network line 31–32–33–34–35 along the edge of the panels VII–II–I–XII and having an abruptly varying stiffness equal to $\frac{1}{2}\lambda$ times the value of K for these respective panels. The beam in question is represented in Fig. 6, together with the diagram of the displacements w and of the moment M_x.

At the net points the beam portions with different stiffnesses are rigidly interconnected (so as to transmit bending). Because of this, the bending moment will have the same magnitude to the left and to the right of each net point; the curvature will undergo an abrupt change there.

To the left of point 33:

$$[M_x]_{33} = -\tfrac{1}{2}\lambda_y K_{II}\varkappa_{II}; \quad \text{so that } \varkappa_{II} = -\frac{[M_x]_{33}}{\frac{1}{2}\lambda_y}\frac{1}{K_{II}}$$

To the right of point 33:

$$[M_x]_{33} = -\tfrac{1}{2}\lambda_y K_{I}\varkappa_{I}; \quad \text{so that } \varkappa_{I} = -\frac{[M_x]_{33}}{\frac{1}{2}\lambda_y}\frac{1}{K_{I}}$$

The relation between the moment and the displacements will now be calculated with the approximation that $\varkappa_I$ and $\varkappa_{II}$ have a constant value over the adjacent beam portions 32–33 and 33–34 respectively.

The moment will then be proportional to the rise of the arc 32–33–34 of the deflection curve, which is equal to:

$$w_{33}-\tfrac{1}{2}(w_{32}+w_{34}) = -\tfrac{1}{2}(w_{32}-2w_{33}+w_{34})$$

Suppose point 33 to be horizontally restrained; then:

$$w_{32} = \tfrac{1}{2}\varkappa_{II}\lambda_x^2; \quad w_{33} = 0; \quad w_{34} = \tfrac{1}{2}\varkappa_{I}\lambda_x^2$$

Therefore:

$$\frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2} = \tfrac{1}{2}[\varkappa_I+\varkappa_{II}]$$

Expressed in the moment at point 33:

$$\frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2} = -\frac{[M_x]_{33}}{\lambda_y}\left[\frac{1}{K_I}+\frac{1}{K_{II}}\right]$$

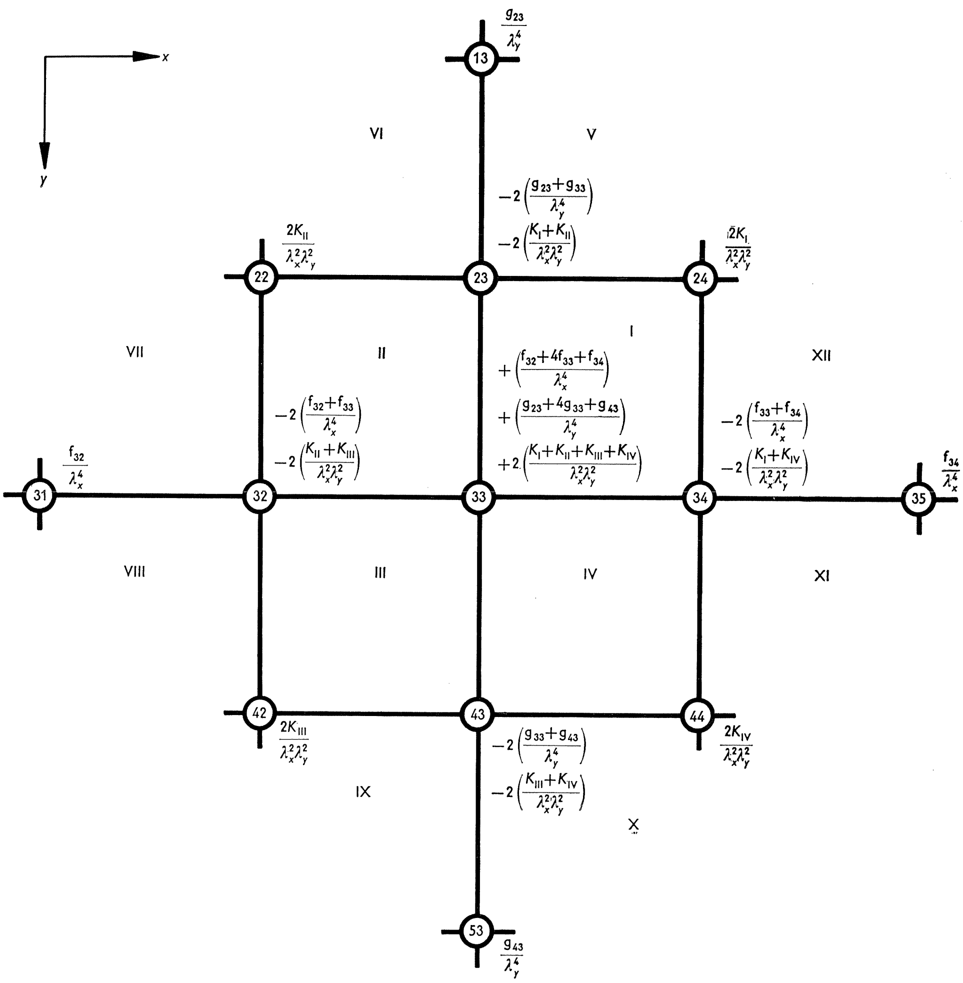


Fig. 7. Coefficients of the difference equation (12a).

Hence:

$$[M_x]_{33} = -\lambda_y \frac{K_{\mathrm{I}}K_{\mathrm{II}}}{K_{\mathrm{I}}+K_{\mathrm{II}}} \times \frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2}$$

The moment at point 33 of the beam along the edge of the panels VIII–III–IV–XI can be determined in the same way. The sum of the two moments at point 33 of the two beams is then:

$$[M_x]_{33} = -\lambda_y \left[\frac{K_{\mathrm{I}}K_{\mathrm{II}}}{K_{\mathrm{I}}+K_{\mathrm{II}}} + \frac{K_{\mathrm{III}}K_{\mathrm{IV}}}{K_{\mathrm{III}}+K_{\mathrm{IV}}}\right] \frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2}$$

The expression between square brackets will from now on be referred to as the 'equivalent stiffness'. To simplify the formulae the following notation is introduced:

$$f_{32} = \frac{K_{\mathrm{VII}}K_{\mathrm{II}}}{K_{\mathrm{VII}}+K_{\mathrm{II}}} + \frac{K_{\mathrm{VIII}}K_{\mathrm{III}}}{K_{\mathrm{VIII}}+K_{\mathrm{III}}} \qquad g_{23} = \frac{K_{\mathrm{II}}K_{\mathrm{VI}}}{K_{\mathrm{II}}+K_{\mathrm{VI}}} + \frac{K_{\mathrm{I}}K_{\mathrm{V}}}{K_{\mathrm{I}}+K_{\mathrm{V}}}$$

$$f_{33} = \frac{K_{\mathrm{II}}K_{\mathrm{I}}}{K_{\mathrm{II}}+K_{\mathrm{I}}} + \frac{K_{\mathrm{III}}K_{\mathrm{IV}}}{K_{\mathrm{III}}+K_{\mathrm{IV}}} \qquad g_{33} = \frac{K_{\mathrm{III}}K_{\mathrm{II}}}{K_{\mathrm{III}}+K_{\mathrm{II}}} + \frac{K_{\mathrm{IV}}K_{\mathrm{I}}}{K_{\mathrm{IV}}+K_{\mathrm{I}}}$$

$$f_{34} = \frac{K_{\mathrm{I}}K_{\mathrm{XII}}}{K_{\mathrm{I}}+K_{\mathrm{XII}}} + \frac{K_{\mathrm{IV}}K_{\mathrm{XI}}}{K_{\mathrm{IV}}+K_{\mathrm{XI}}} \qquad g_{43} = \frac{K_{\mathrm{IX}}K_{\mathrm{III}}}{K_{\mathrm{IX}}+K_{\mathrm{III}}} + \frac{K_{\mathrm{X}}K_{\mathrm{IV}}}{K_{\mathrm{X}}+K_{\mathrm{IV}}}$$

Then the moments occurring in equation (6) will be:

$$\left.\begin{aligned}
[M_x]_{32} &= -\lambda_y f_{32} \frac{w_{31}-2w_{32}+w_{33}}{\lambda_x^2} \\
[M_x]_{33} &= -\lambda_y f_{33} \frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2} \\
[M_x]_{34} &= -\lambda_y f_{34} \frac{w_{33}-2w_{34}+w_{35}}{\lambda_x^2} \\
[M_y]_{43} &= -\lambda_x g_{43} \frac{w_{53}-2w_{43}+w_{33}}{\lambda_y^2} \\
[M_y]_{33} &= -\lambda_x g_{33} \frac{w_{43}-2w_{33}+w_{23}}{\lambda_y^2} \\
[M_y]_{23} &= -\lambda_x g_{23} \frac{w_{33}-2w_{23}+w_{13}}{\lambda_y^2}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (7a)$$

Substitution of equation (7a) into equation (6) gives:

$$\left.\begin{aligned}
R_b = {} & \lambda_y f_{32} \frac{w_{31}-2w_{32}+w_{33}}{\lambda_x^3} - 2\lambda_y f_{33} \frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^3} \\
& + \lambda_y f_{34} \frac{w_{33}-2w_{34}+w_{35}}{\lambda_x^3} + \lambda_x g_{43} \frac{w_{53}-2w_{43}+w_{33}}{\lambda_y^3} \\
& - 2\lambda_x g_{33} \frac{w_{43}-2w_{33}+w_{23}}{\lambda_y^3} + \lambda_x g_{23} \frac{w_{33}-2w_{23}+w_{13}}{\lambda_y^3}
\end{aligned}\right\} \quad \ldots \quad (8a)$$

The reactions produced at the net points by the torsional panels are determined in the same manner as has been described with reference to plates of constant thickness in Chapter I. Each torsional panel produces a reaction P equal to twice the stiffness multiplied by the twist of the panel concerned (equation 10).

The total reaction produced at point 33 is:

$$R_w = 2K_{\mathrm{I}} \frac{w_{33} - w_{34} + w_{24} - w_{23}}{\lambda_x \lambda_y} + 2K_{\mathrm{II}} \frac{w_{33} - w_{23} + w_{33} - w_{32}}{\lambda_x \lambda_y}$$

$$+ 2K_{\mathrm{III}} \frac{w_{33} - w_{32} + w_{42} - w_{43}}{\lambda_x \lambda_y} + 2K_{\mathrm{IV}} \frac{w_{33} - w_{43} + w_{44} - w_{34}}{\lambda_x \lambda_y} \quad \ldots \text{(11a)}$$

Substitution of equations (8a) and (11a) into equation (5) gives the difference equation for point 33:

$$f_{32} \frac{w_{31} - 2w_{32} + w_{33}}{\lambda_x^4} - 2f_{33} \frac{w_{32} - 2w_{33} + w_{34}}{\lambda_x^4}$$

$$+ f_{34} \frac{w_{33} - 2w_{34} + w_{35}}{\lambda_x^4} + g_{43} \frac{w_{53} - 2w_{43} + w_{33}}{\lambda_y^4}$$

$$- 2g_{33} \frac{w_{43} - 2w_{33} + w_{23}}{\lambda_y^4} + g_{23} \frac{w_{33} - 2w_{23} + w_{13}}{\lambda_y^4}$$

$$+ 2K_{\mathrm{I}} \frac{w_{33} - w_{34} + w_{24} - w_{23}}{\lambda_x^2 \lambda_y^2} + 2K_{\mathrm{II}} \frac{w_{33} - w_{23} + w_{22} - w_{32}}{\lambda_x^2 \lambda_y^2}$$

$$+ 2K_{\mathrm{III}} \frac{w_{33} - w_{32} + w_{42} - w_{43}}{\lambda_x^2 \lambda_y^2} + 2K_{\mathrm{IV}} \frac{w_{33} - w_{43} + w_{44} - w_{34}}{\lambda_x^2 \lambda_y^2} = q \quad \ldots \text{(12a)}$$

The first six terms of equation (12a) are flexural terms; the others are torsional terms. In Fig. 7 the coefficients of equation (12a) are represented in the form of an array of differences.

It is an obvious suggestion to check the validity of equation (12a) for some particular plate shapes. In the case of a flat plate (constant thickness) the equivalent stiffnesses f and g are equal to K, so that equation (12a) becomes identical with equation (12) for such a plate.

For a plate whose difference panels alternately have a stiffness K and zero stiffness in checkerboard fashion the equivalent stiffnesses f and g are zero. The load could in that case be transmitted only by the torsional resistance of the panels. In general this is possible only for certain special loads and/or boundary conditions.

A rectangular plate with the above mentioned checkerboard pattern and having two opposite edges unsupported can be rolled up in the form of a cylinder, i.e., any deformation not associated with torsion can freely occur. This phenomenon can be verified with a simple little test, which also constitutes a justification of the choice of the analogy model. In point of fact only the beams which are in a direct line with each other are rigidly interconnected, so that in this case the beams cannot transmit bending.

The remaining flexural stiffness of the plate with checkerboard pattern is so small that it can be assumed that the equivalent beams such as 33–34 of panel I and 32–33 of panel III are not rigidly interconnected.

In Chapter 1 it was already noted that for the points located at a distance of one mesh width from the edge of the plate, and for the edge points themselves, the difference equation has to be modified, depending on the nature of the boundary condition. In establishing the difference equation for these points it must be taken into account that one or more of the adjacent points to be considered are located outside the plate.

In the existing literature this problem is solved by giving these 'external points' a judiciously chosen displacement appropriate to the relevant boundary conditions. Since the number of unknown external points is equal to the number of boundary conditions, the displacements to be applied to these points are uniquely determined. For example, for a restrained (rigidly fixed) edge the displacements given to the external points located at a distance of one mesh width from the edge are equal to the displacements of the internal points located at the same distance from the edge. For more complicated boundary conditions, such as those at unsupported edges and corners, expressing the displacements of the external points in those of the internal points is a rather laborious task.

A quicker result is obtained by making use of the difference equation – derived in the foregoing – for plates with abrupt changes in stiffness (equation 12a). This equation in fact directly yields a difference equation for the points at the edge and at a distance of one mesh width from edge. It does this without having recourse to external points. Three boundary conditions will now be considered as examples:

a. *Restrained edge*

In Fig. 5 the network line 24–34–44 will be assumed to be a restrained edge (rigidly fixed), with the plate situated to the left thereof.

To the right of this network line all the stiffnesses are infinitely large, while the displacements of the point on this line are zero. The difference equation for point 33 can now be determined as follows:

For otherwise constant plate thickness the equivalent stiffnesses f and g are:

$$f_{32} = K; f_{33} = K; f_{34} = 2K$$

$$g_{23} = K;\ g_{33} = K;\ g_{43} = K$$

Substitution of these stiffnesses into the array of Fig. 7 yields the array according to Fig. 8a for a square-meshed network ($\lambda_x = \lambda_y = \lambda$).

If the displacements of the edge are to have prescribed values, then the array represented in Fig. 8b is obtained.

b. *Unsupported edge*

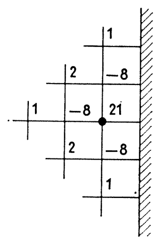
It will be assumed that the network line 13–23–33–43–53 in Fig. 5 represents an unsupported edge, the plate being to the left thereof. To the right of this line all the stiffnesses are zero.

The difference equation for point 33 can now be determined as follows: For constant plate thickness the equivalent stiffnesses f and g are:

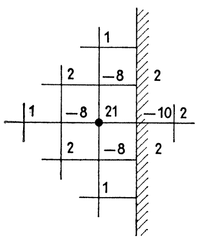
$$f_{32} = K; \quad f_{33} = 0; \quad f_{34} = 0$$

$$g_{23} = \tfrac{1}{2}K; \; g_{33} = \tfrac{1}{2}K; \; g_{43} = \tfrac{1}{2}K$$

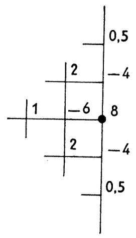
Substitution of these stiffnesses into the array of Fig. 7 yields the array according to Fig. 8c for a square-meshed network ($\lambda_x = \lambda_y = \lambda$).



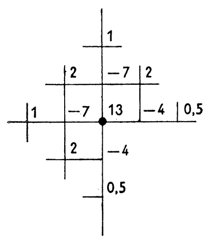
a. Restraint $K \Sigma w = q\lambda^4$.



b. Restraint $K \Sigma w = q\lambda^4$.



c. Unsupported edge $K \Sigma w = \frac{1}{2}q\lambda^4$.



d. Unsupported re-entrant angle $K \Sigma w = \frac{3}{4}q\lambda^4$.

Fig. 8. Difference arrays for points in the vicinity of edges for a square-meshed difference network ($\lambda_x = \lambda_y = \lambda$).

c. *Unsupported re-entrant angle*

In Fig. 5 the angle 43–33–34 is assumed to be a re-entrant angle where two unsupported edges of the plate intersect.

The difference equation for point 33 can be determined from the general equation by supposing a plate of zero stiffness to be present within the right angle.

For constant plate thickness the equivalent stiffnesses f and g are:

$$f_{32} = K; f_{33} = \tfrac{1}{2}K; f_{34} = \tfrac{1}{2}K$$

$$g_{23} = K; \; g_{33} = \tfrac{1}{2}K; \; g_{43} = \tfrac{1}{2}K$$

Substitution of these stiffnesses into the array of Fig. 7 yields the array according to Fig. 8d for a square-meshed network ($\lambda_x = \lambda_y = \lambda$).

From the foregoing it will be apparent that the general difference equation (12a) can be used for writing down the difference equations for all points of the plate. For those panels of the difference network which coincide with, for instance, an aperture or an infinitely stiff part of the plate it is merely necessary to insert $K = 0$, or $K = \infty$ into the equation.

Thus, for example, plates of constant thickness with apertures constitute a special case of the plate with abrupt changes in thickness.

When difference equations have in this way been established for all the points of the plate determined by the difference network, the problem has been reduced to the solving of a set of linear equations expressed in the displacements. The matrix of the coefficients of this set of equations has some special properties which, on the one hand, provide a check on the calculated coefficients and which, on the other hand, can be used in solving the set.

In the first place it is to be noted that the sum of the coefficients in equation (12a) (see also Fig. 7) is zero because the sum of the coefficients of each part constituting equation (12a) is zero.

In physical terms this can be explained by considering that, in the case of a rigid vertical displacement $\bar{w}$ of an unloaded plate, the plate itself will offer no resistance to such displacement. Now if for each w in equation (12a) the value $\bar{w}$ is substituted, only the right-hand member will be zero ($q \neq 0$) if the sum of the coefficients is zero.

Also in the case of rigid rotation about an arbitrary line, with a linear relation between the displacements, the right-hand member of equation (12a) must be zero. For the matrix this means that there exists a relation between the coefficients of each row.

Both properties provide a check on the magnitude of the coefficients.

The same argument is applicable to points located beside or at unsupported edges; for this reason the same checks are valid also for the relevant row of the matrix of coefficients. However, for points located beside restrained edges or edges with hinged bearings, the same is not applicable, if the condition that the displacement at the edge is zero has already been utilised.

If a prescribed deformation of the edge is accepted as a permissible condition, then a somewhat different array of differences will be obtained, for which the checks in question will indeed be valid. In that case, for a restrained edge the array represented in Fig. 8b instead of that in Fig. 8a should be used.

Another important property is the reciprocity relationship which exists between the coefficients in the equations for various net points. Let a_{ij} denote the coefficient of the displacement w_j in the equation for point i and let a_{ji} denote the coefficient of the displacement w_i in the equation for point j; then the reciprocity relationship in question is expressed by $a_{ij} = a_{ji}$.

This property can be derived with the aid of Maxwell's theorem. The consequence of this is that the matrix of coefficients is symmetric about the principal diagonal, provided that also for those points which are affected by the boundary conditions, the difference equations are determined with the aid of equation (12a).

For the solving of linear equations having a symmetric matrix of the coefficients with the aid of an electronic computer, special programmes can be employed which require less machine time and which, for a given memory capacity, enable larger sets of equations to be dealt with than programmes for equations having an arbitrary matrix. Besides, the relatively large number of noughts in the matrix may be an advantageous feature if solution programmes are used which utilise this property.

As it has proved possible to employ equation (12a) for all points of the difference network, including the points in the vicinity of the edge and at the edge itself, it may be worth while to make a special programme for this.

The moments and shear forces are determined by a procedure similar to that described with reference to plates of constant thickness in Chapter 1.

This will be demonstrated with regard to the bending moment for point 33 of Fig. 5.

At the side of panels I and II the following holds true:

$$[m_x]_{33} = -2\,\frac{K_{\mathrm{I}}K_{\mathrm{II}}}{K_{\mathrm{I}}+K_{\mathrm{II}}}\,\frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2}$$

And at the side of panels III and IV: . . . . (13a)

$$[m_x]_{33} = -2\,\frac{K_{\mathrm{III}}K_{\mathrm{IV}}}{K_{\mathrm{III}}+K_{\mathrm{IV}}}\,\frac{w_{32}-2w_{33}+w_{34}}{\lambda_x^2}$$

Here the distributed moments which are valid for a zone of width $\frac{1}{2}\lambda_y$ beside the network line have been directly written down.

The torsional moment in, for example, panel 33–32–22–23 is:

$$[m_{xy}]_{\mathrm{II}} = -K_{\mathrm{II}}\,\frac{w_{33}-w_{32}+w_{22}-w_{23}}{\lambda_x\lambda_y} \qquad (14a)$$

Just as was done in the treatment of slabs with constant thickness, the torsional share of the shear force will first be considered more closely.

In plates with variable thickness the concentrated shear forces will likewise partly cancel one another along the edge that the panels share in common. Now, however, only a part of the difference of the concentrated shear forces is uniformly distributed; the rest continues to exist as a concentrated shear force.

It can be demonstrated that in the most general case – if, for instance, $K_{III} > K_{II}$ – there exists in the panels II and III a uniformly distributed shear force in the *x*-direction whose magnitude is:

$$q_x = \tfrac{1}{2}\,\frac{\dfrac{K_{II}}{K_{III}} \times P_{III} - P_{II}}{\lambda_y}$$

and in the panel with stiffness K_{III} there exists along the common edge a concentrated shear force of the following magnitude:

$$R_x = \tfrac{1}{2}\,\frac{K_{III} - K_{II}}{K_{III}}\,P_{III}$$

To clarify this, three limiting cases will be considered:

a. If $K_{II} = K_{III}$, the concentrated shear force does not occur, and the uniformly distributed shear force in the two panels is:

$$q_x = \tfrac{1}{2}\,\frac{P_{III} - P_{II}}{\lambda_y}$$

b. If $K_{II} = 0$, there is only a concentrated shear force at the edge of panel III:

$$R_x = \tfrac{1}{2}\,P_{III}$$

c. If $K_{III} > K_{II}$ and the two panels have the same tilt, then there is no uniformly distributed shear force, but in the panel with stiffness K_{III} there exists along the common edge a concentrated shear force:

$$R_x = \tfrac{1}{2}P_{III} - \tfrac{1}{2}P_{II}$$

The shear forces in the beams of the analogy model are, converted for the plate, uniformly distributed over a zone of width $\tfrac{1}{2}\lambda$ beside the network line.

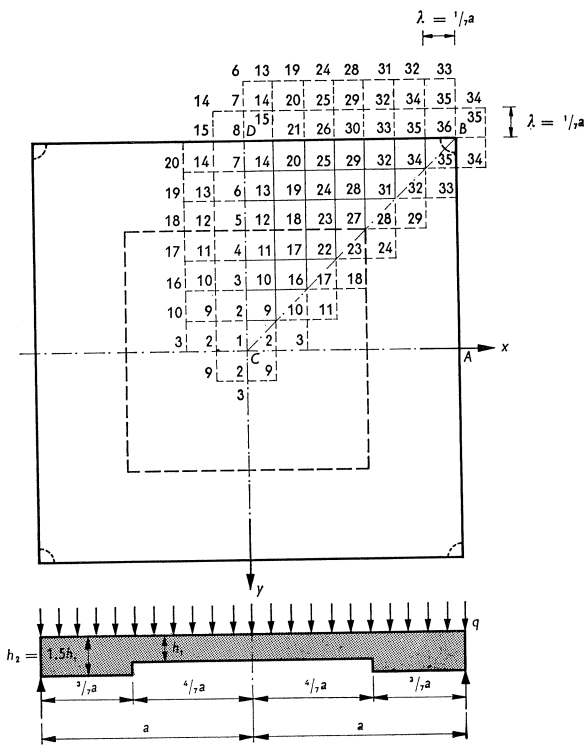


Fig. 9.
Square ideal internal panel of a strip floor supported on columns.

Hence, if $K_{III} > K_{II}$, the uniformly distributed shear force in the x-direction on the length 32–33 at the side of panel II will be:

$$q_x = \frac{[m_x]_{33} - [m_x]_{32}}{\lambda_x} + \tfrac{1}{2}\,\frac{\dfrac{K_{II}}{K_{III}} \times P_{III} - P_{II}}{\lambda_y}$$

$$= \frac{[m_x]_{33} - [m_x]_{32}}{\lambda_x} + \frac{\dfrac{K_{II}}{K_{III}}\,[m_{xy}]_{III} - [m_{xy}]_{II}}{\lambda_y}$$

and on the length 32–33 at the side of panel III:

$$q_x = \frac{[m_x]_{33} - [m_x]_{32}}{\lambda_x} + \frac{\dfrac{K_{II}}{K_{III}}\,[m_{xy}]_{III} - [m_{xy}]_{II}}{\lambda_y} \qquad (15a)$$

Furthermore a concentrated force acts along the edge of panel III:

$$R_x = \tfrac{1}{2}\,\frac{K_{III} - K_{II}}{K_{III}}\,P_{III}$$

$$= \frac{K_{III} - K_{II}}{K_{III}}\,[m_{xy}]_{III}$$

3 Worked example

The theory which has been presented in the foregoing will now be further elucidated with the aid of an example relating to the analysis of the forces acting in a 'strip floor'. This type of floor constitutes an intermediate case between flat slab floors and floors supported on beams which span from column to column. The essential difference is that the structural behaviour of the beam-and-slab floor is such that the beams and slabs can, with satisfactory approximation, be analysed separately, whereas in the strip floor (comprising thickened strips extending from column to column) the floor structure must be considered as a whole. Also, in the strip floor the column heads will, generally speaking, not be (or be only slightly) enlarged or flared, since thickened strips are provided. The actual shape of the column head can be ignored, so that analysis by means of the method of differences is possible [5].

Fig. 9 represents a square ideal internal panel subjected to a uniformly distributed loading q. By 'ideal internal panel' is to be understood a panel which is located at such a distance from the edge of a floor of this kind that, for uniformly distributed loading, the lines connecting the columns can be regarded as lines of symmetry for the deflection surface.

The thickness of the slab h_1 is two-thirds of the thickness of the strip h_2, so that – if K denotes the stiffness of the central part – the stiffness of the thickened strip is $(h_2/h_1)^3 K = 3.375\, K$. The width of the strip is 3/7 of the centre-to-centre spacing of the columns.

Because of the symmetry, it is necessary only to consider a half quadrant (namely, the triangle BCD) of the square panel. The slab is provided with a square-meshed difference network with a mesh width $\lambda = a/7$.

The width of the slab and the width of the strips have been so chosen that the transition from slab to strip coincides with a network line. The column supporting the slab is conceived as a 'point bearing', so that the condition applicable at point 36 is: $w = 0$. For the other points along the three boundary lines of the octant under consideration the boundary conditions (slope and shear force at right angles to the said boundary lines are zero) are satisfied by extending the network symmetrically outwards beyond the three symmetry lines (see Fig. 9).

For each point of the slab the equation (12a) is valid. In all, this yields a set of 35 equations for the 35 unknown displacements w_1 to w_{35} ($w_{36} = 0$).

When these equations have been solved, the moments and shear forces can be determined from the calculated displacements.

The way in which the equations are established will now, by way of example, be shown for the points 1, 5 and 27.

Point 1: Equation (4) is valid for this point

$$20w_1 - 8[w_2+w_2+w_2+w_2] + 2[w_9+w_9+w_9+w_9] + [w_3+w_3+w_3+w_3] = \frac{q\lambda^4}{K}$$

or:

$$20w_1 - 32w_2 + 8w_9 + 4w_3 = \frac{q\lambda^4}{K}$$

To obtain a symmetric matrix, the equation is multiplied by 1/8. This is associated with the fact that only one-eighth of the slab is being considered.

$$2.5w_1 - 4w_2 + w_9 + 0.5w_3 = 0.125\,\frac{q\lambda^4}{K} \quad \ldots\ldots\ldots\ldots \quad (\bar{w}_1)$$

Point 5: Equation (12a) is valid for this point

The equivalent stiffnesses are:

$$f_5 = f_{12} = \frac{3.375K \times 3.375K}{3.375K + 3.375K} + \frac{K \times K}{K+K} = 2.187500K$$

$$g_5 = \frac{K \times 3.375K}{K+3.375K} + \frac{K \times 3.375K}{K+3.375K} = 1.542857K$$

$$g_4 = K$$

$$g_6 = 3.375K$$

Therefore:

$$2.187500[w_{18}-2w_{12}+w_5]-2\times 2.187500[w_{12}-2w_5+w_{12}]+$$
$$+2.187500[w_5-2w_{12}+w_{18}]+$$
$$+1\times[w_3-2w_4+w_5]-2\times 1.542857[w_4-2w_5+w_6]+3.375[w_5-2w_6+w_7]+$$
$$+2\times 3.375[w_5-w_{12}+w_{13}-w_6]+2\times 3.375[w_5-w_6+w_{13}-w_{12}]+$$
$$+2\times 1[w_5-w_{12}+w_{11}-w_4]+2\times 1[w_5-w_4+w_{11}-w_{12}] = \frac{q\lambda^4}{K}$$

To obtain a symmetric matrix, this equation as a whole is multiplied by $\frac{1}{2}$.
On rearranging:

$$0.5w_3-4.542857w_4+20.585714w_5-11.667857w_6+$$
$$+1.6875w_7+2w_{11}-17.5w_{12}+6.75w_{13}+2.1875w_{18} = 0.5\frac{q\lambda^4}{K} \quad \ldots\ldots \quad (\bar{w}_5)$$

Point 27: Equation (12a) is valid for this point also

$$f_{23} = g_{23} = \frac{3.375K\times 3.375K}{3.375K+3.375K}+\frac{K\times K}{K+K} = 2.187500K$$

$$f_{27} = g_{27} = \frac{3.375K\times 3.375K}{3.375K+3.375K}+\frac{K\times 3.375K}{K+3.375K} = 2.458929K$$

$$f_{28} = g_{28} = 3.375K$$

Therefore:

$$3.375[w_{29}-2w_{28}+2w_{27}]-2\times 2.458929[w_{28}-2w_{27}+w_{23}]+$$
$$+2.1875[w_{27}-2w_{23}+w_{18}]+2.1875[w_{18}-2w_{23}+w_{27}]-$$
$$-2\times 2.458929[w_{23}-2w_{27}+w_{28}]+3.375[w_{27}-2w_{28}+w_{29}]+$$
$$+2\times 3.375[w_{27}-w_{23}+w_{24}-w_{28}]+2\times 3.375[w_{27}-w_{28}+w_{31}-w_{28}]+$$
$$+2\times 3.375[w_{27}-w_{28}+w_{24}-w_{23}]+2\times[w_{27}-w_{23}+w_{22}-w_{23}] = \frac{q\lambda^4}{K}$$

On multiplying this equation by $\frac{1}{2}$ for the same reason and rearranging the terms:

$$2.1875w_{18}+w_{22}-18.042858w_{23}+6.75w_{24}+26.523216w_{27}-$$
$$-25.167858w_{28}+3.375w_{29}+3.375w_{31} = 0.5\frac{q\lambda^4}{K} \quad \ldots\ldots\ldots\ldots \quad (\bar{w}_{27})$$

All the equations can be built up in this way. The set of equations is set down in Table I (see pp. 22 and 23).

The solution of these equations, as determined with the aid of an electronic computer, is given in Table II.

Table I. (first part)

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9
1	2.5	−4	0.5						1
2	−4	12.5	−4	0.5					−8
3	0.5	−4	10	−4	0.5				2
4		0.5	−4	10.271429	−4.542857	0.771428			
5			0.5	−4.542857	20.585714	−11.667857	1.6875		
6				0.771428	−11.667857	32.833929	−13.50	1.6875	
7					1.6875	−13.5	35.4375	−13.5	
8						1.6875	−13.5	1.6875	
9	1	−8	2						11
10		3	−8	2					−8
11			2	−8	2				1
12				2	−17.5	6.75			
13					6.75	−27	6.75		
14						6.75	−27	6.75	
15							6.75	−13.5	
16			1						1
17				1					
18					2.1875				
19						3.375			
20							3.375		
21								1.6875	
22									
23									
24									
25									
26									
27									
28									
29									
30									
31									
32									
33									
34									
35									

Table I. (second part)

	w_{10}	w_{11}	w_{12}	w_{13}	w_{14}	w_{15}	w_{16}	w_{17}	w_{18}
1									
2	3								
3	−8	2					1		
4	2	−8	2					1	
5		2	−17.5	6.75					2.1875
6			6.75	−27	6.75				
7				6.75	−27	6.75			
8					6.75	−13.5			
9	−8	1					1		
10	23	−8	1				−8	3	
11	−8	21.542857	−9.085714	1.542857			2	−8	2
12	1	−9.085714	43.358928	−23.335714	3.375			2	−17.5
13		1.542857	−23.335714	69.042857	−27	3.375			6.75
14			3.375	−27	74.25	−27			
15				3.375	−27	35.4375			
16	−8	2					10	−8	1
17	3	−8	2				−8	22.542857	−9.085714
18		2	−17.5	6.75			1	−9.085714	41.171428
19			6.75	−27	6.75			1.542857	−23.335714
20				6.75	−27	6.75			3.375
21					6.75	−13.5			
22		1					1	−8	2
23			2.1875					3	−17.5
24				3.375					6.75
25					3.375				
26						1.6875			
27									2.1875
28									
29									
30									
31									
32									
33									
34									
35									

Table I. (third part)

	w_{19}	w_{20}	w_{21}	w_{22}	w_{23}	w_{24}	w_{25}	w_{26}	w_{27}
1									
2									
3									
4									
5									
6	3.375								
7		3.375							
8			1.6875						
9									
10									
11				1					
12	6.75				2.1875				
13	−27	6.75				3.375			
14	6.75	−27	6.75				3.375		
15		6.75	−13.5					1.6875	
16				1					
17	1.542857			−8	3				
18	−23.335714	3.375		2	−17.5	6.75			2.1875
19	65.667857	−27	3.375		6.75	−27	6.75		
20	−27	70.875	−27			6.75	−27	6.75	
21	3.375	−27	33.75				6.75	−13.5	
22				10.542857	−9.085714	1.542857			1
23	6.75			−9.085714	43.442853	−23.335714	3.375		−18.042858
24	−27	6.75		1.542857	−23.335714	65.667857	−27	3.375	6.75
25	6.75	−27	6.75		3.375	−27	70.875	−27	
26		6.75	−13.5			3.375	−27	33.75	
27				1	−18.042858	6.75			26.523216
28	3.375				9.208933	−27	6.75		−25.167858
29		3.375				6.75	−27	6.75	3.375
30			1.6875				6.75	−13.5	
31						3.375			3.375
32							3.375		
33								1.6875	
34									
35									

Table I. (fourth part)

	w_{28}	w_{29}	w_{30}	w_{31}	w_{32}	w_{33}	w_{34}	w_{35}	$q\lambda^4/k$
1									0.125
2									0.50
3									0.50
4									0.50
5									0.50
6									0.50
7									0.50
8									0.25
9									0.5
10									1
11									1
12									1
13									1
14									1
15									0.5
16									0.5
17									1
18									1
19	3.375								1
20		3.375							1
21			1.6875						0.5
22									0.5
23	9.208933								1
24	−27	6.75		3.375					1
25	6.75	−27	6.75		3.375				1
26		6.75	−13.5			1.6875			0.5
27	−25.167858	3.375		3.375					0.5
28	73.333925	−27	3.375	−27	10.125				1
29	−27	70.875	−27	6.75	−27	6.75	3.375		1
30	3.375	−27	33.75		6.75	−13.5		1.6875	0.5
31	−27	6.75		33.75	−27	3.375	3.375		0.5
32	10.125	−27	6.75	−27	77.625	−27	−27	10.125	1
33		6.75	−13.5	3.375	−27	33.75	6.75	−13.5	0.5
34		3.375		3.375	−27	6.75	37.125	−27	0.5
35			1.6875		10.125	−13.5	−27	42.1875	0.5

Table II. Displacements in $q\lambda^4/K$ $(\lambda = 1/7a)$

$w_1 = 83.2289$	$w_{13} = 59.8927$	$w_{25} = 42.3732$
$w_2 = 81.8197$	$w_{14} = 56.5443$	$w_{26} = 40.7655$
$w_3 = 77.8793$	$w_{15} = 55.3466$	$w_{27} = 44.5941$
$w_4 = 72.2690$	$w_{16} = 72.2321$	$w_{28} = 37.3359$
$w_5 = 66.4242$	$w_{17} = 66.2487$	$w_{29} = 31.8817$
$w_6 = 61.7240$	$w_{18} = 59.8976$	$w_{30} = 29.7643$
$w_7 = 58.4810$	$w_{19} = 54.6436$	$w_{31} = 28.2032$
$w_8 = 57.3246$	$w_{20} = 50.9553$	$w_{32} = 20.9734$
$w_9 = 80.3921$	$w_{21} = 49.6216$	$w_{33} = 17.9229$
$w_{10} = 76.3961$	$w_{22} = 59.7801$	$w_{34} = 11.5967$
$w_{11} = 70.6940$	$w_{23} = 52.7368$	$w_{35} = 6.9448$
$w_{12} = 64.7264$	$w_{24} = 46.7134$	$w_{36} = 0$

Next, the moments (equations 13a, 14a) and the shear forces (equation 15a) can be determined from the calculated displacements.

To illustrate this, some more examples will now be given.

In the strip at point 5 (equation 13a):

$$m_x = -3.375K\,\frac{w_{12}-2w_5+w_{12}}{\lambda^2} = 11.4601q\lambda^2 = 0.2338qa^2$$

In the slab at point 5:

$$m_x = -K\,\frac{w_{12}-2w_5+w_{12}}{\lambda^2} = 3.3956q\lambda^2 = 0.0693qa^2$$

At the transition of strip and slab at point 5:

$$m_y = -2\times\frac{K\times 3.375K}{K+3.375K}\,\frac{w_4-2w_5+w_6}{\lambda^2} = -1.7660q\lambda^2 = -0.0361qa^2$$

At the transition of slab and strip at point 27:

$$m_x = -2\times\frac{K\times 3.375K}{K+3.375K}\,\frac{w_{23}-2w_{27}+w_{28}}{\lambda^2} = -1.3647q\lambda^2 = -0.0278qa^2$$

In the panel 12–5–6–13 (equation 14a):

$$m_{xy} = -3.375K\,\frac{w_{12}-w_5+w_6-w_{13}}{\lambda^2} = -0.4505q\lambda^2 = -0.0092qa^2$$

In the panel 11–4–5–12:

$$m_{xy} = -K\,\frac{w_{11}-w_4+w_5-w_{12}}{\lambda^2} = -0.1228q\lambda^2 = -0.0025qa^2$$

In the strip extending between points 12 and 5 (equation 15a):

$$q_x = \frac{[m_x]_{12}-[m_x]_5}{\lambda} + \frac{-\dfrac{1}{3.375}[m_{xy}]_{\text{strip}}+[m_{xy}]_{\text{slab}}}{\lambda}$$
$$= [10.5667-11.4601+0.1335-0.1228]$$
$$= -0.8827q\lambda = -0.1261qa$$

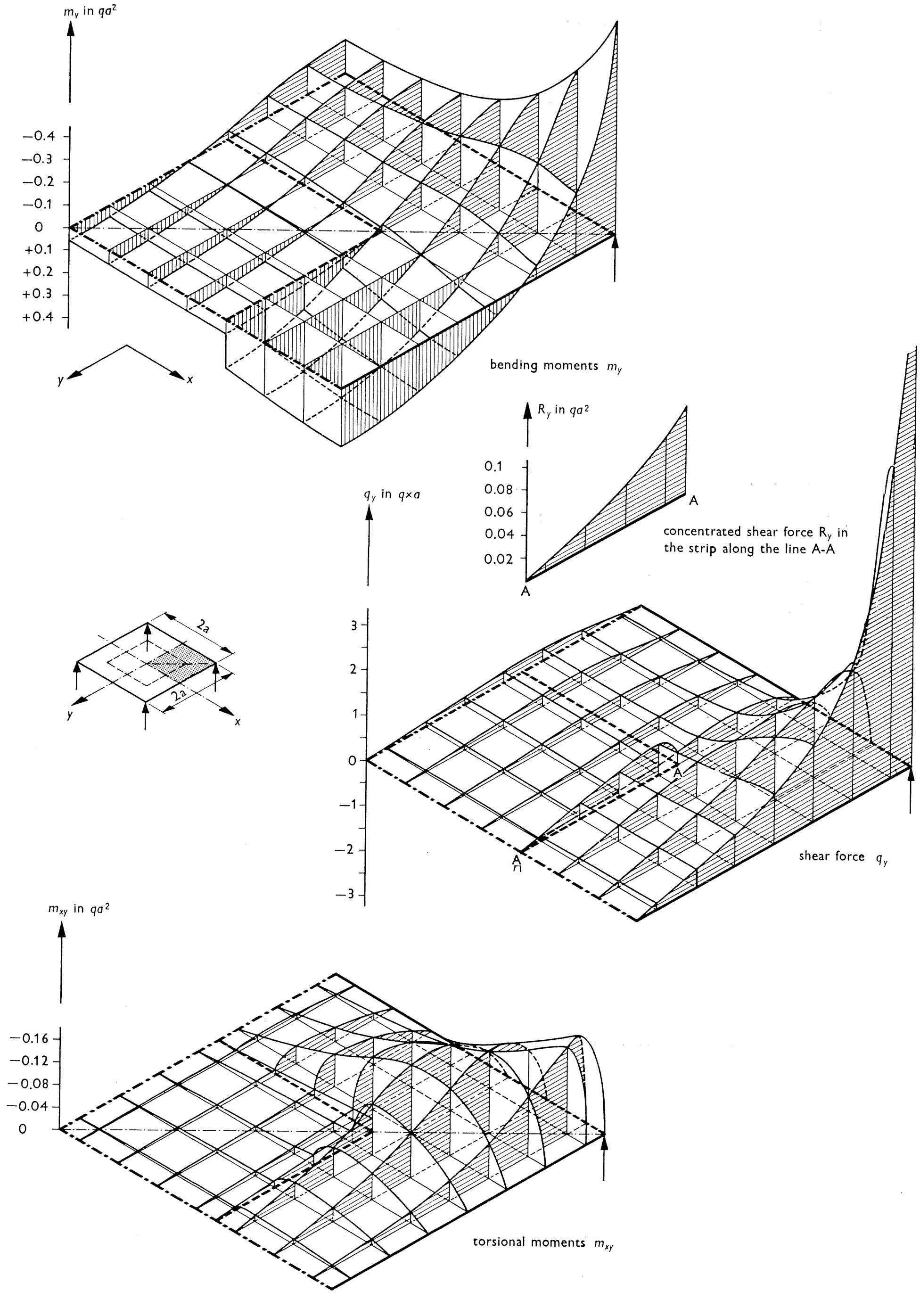

m_y in qa²
−0.4
−0.3
−0.2
−0.1
0
+0.1
+0.2
+0.3
+0.4
y
x
bending moments m_y
R_y in qa²
0.1
0.08
0.06
0.04
0.02
A
A
concentrated shear force R_y in the strip along the line A-A
q_y in q×a
3
2
1
0
−1
−2
−3
2a
2a
y
x
A
A
shear force q_y
m_xy in qa²
−0.16
−0.12
−0.08
−0.04
0
torsional moments m_xy

Fig. 10.

In the slab extending between points 12 and 5:

$$q_x = \frac{[m_x]_{12} - [m_x]_5}{\lambda} + \frac{-\dfrac{1}{3.375}[m_{xy}]_{\text{strip}} + [m_{xy}]_{\text{slab}}}{\lambda}$$

$$= [3.1309 - 3.3956 + 0.1335 - 0.1228]$$

$$= -0.2540 q\lambda = -0.0363 qa$$

The concentrated shear force along the edge of the strip between points 12 and 5 is:

$$R_x = -\frac{3.375-1}{3.375}[m_{xy}]_{\text{strip}}$$

$$= -\frac{2.375}{3.375} \times 0.4505 = -0.317 q\lambda = 0.0065 qa^2$$

The distribution of the moments and shear forces is shown in Fig. 10.

A preliminary indication as to the reliability of the method can be obtained by carrying out a number of equilibrium checks.

For the quadrant ABCD in Fig. 9 the requirement is that the following equilibrium condition must be satisfied:

$$-\sum_A^B m_x + \sum_D^C m_x = 49 q\lambda^2 \times 3.5\lambda = 171.5 q\lambda^3$$

since the torsional moments and shear forces must – for reasons of symmetry – be zero along the edges of the quadrant.

From summation of the moments it follows that: $-\sum_A^B m_x = 121.5639$

$$+\sum_D^C m_x = 49.9166$$

total 171.4805

The relative error is only $0.12^0/_{00}$.

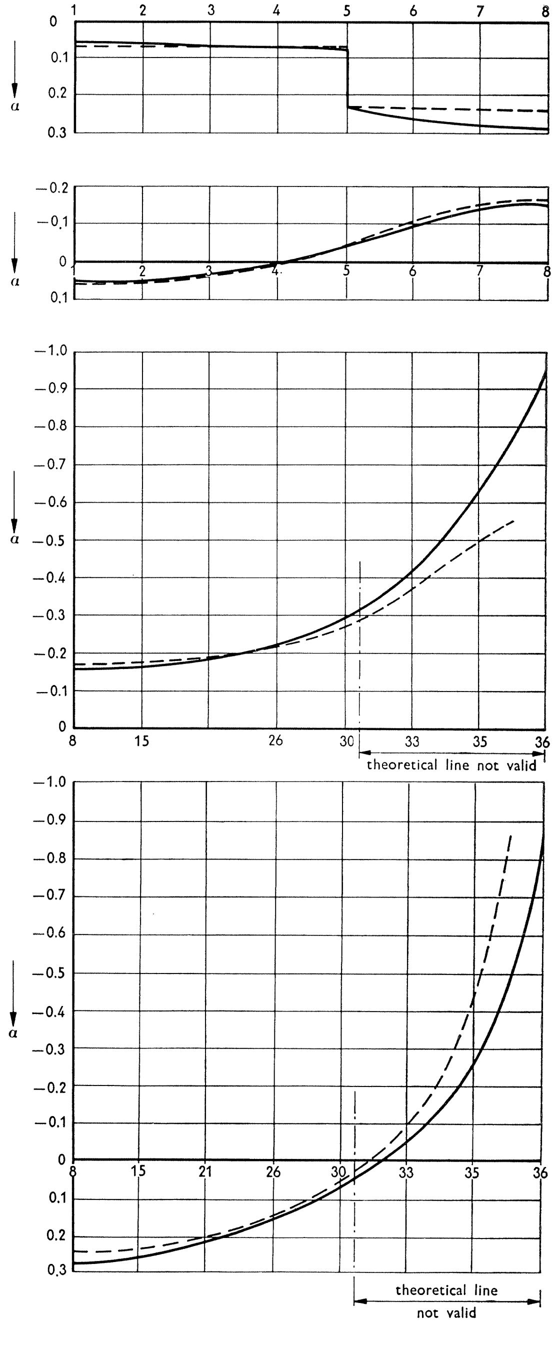
Other equilibrium checks, more particularly those for shear force, also yielded very satisfactory results.

4 Model test

With a view to verifying the analysis, a Perspex model was made in which the moments were determined with the aid of the moiré method.

Some results are given in Fig. 11.

As already stated, Poisson's ratio was assumed to be zero ($\nu = 0$) in the analysis. For this reason there cannot be complete agreement with the results of



M_x along the line 1–2–3–4–5–6–7–8 in $\alpha q a^2$ (transverse moment in middle strip)

M_y along the line 1–2–3–4–5–6–7–8 in $\alpha q a^2$ (longitudinal moment in middle strip)

M_y along the line 8–15–21–26–30–33–35–36 in $\alpha q a^2$ (transverse moment in column strip)

M_x along the line 8–15–21–26–30–33–35–36 in $\alpha q a^2$ (longitudinal moment in column strip)

Fig. 11. Comparison of the calculated moments (———) and the moments determined from the model tests (– – – –).

the model test, since for Perspex $\nu \approx 0.4$. From the curvatures determined by means of the moiré method the moments were calculated with $\nu = 0$, so that actually it is not the moments but the curvatures that are compared.

Except in the vicinity of the column, there can be said to be very good agreement between the calculated values and the results obtained from the model test.

Because of the relatively large dimension of the mesh width of the difference network, the moments in the vicinity of the column are not accurately represented. In addition, the actual dimensions of the support have not been taken into account. Hence it is necessary to carry out a supplementary calculation for determining the pattern of forces acting in the vicinity of the support. For this the relevant literature should be consulted [5, 8, 9].

5 Concluding remarks

From the foregoing treatment of the subject it appears that the calculus of differences offers good possibilities for the analysis of plates (slabs) of abruptly varying stiffness and with arbitrary boundary conditions. As compared with the grid framework method, in which both the displacement and the slope in two mutually perpendicular directions at the net points are taken to be unknown, the present method has the advantage that only one-third of the number of equations have to be solved. The advantage claimed for the grid framework method over the difference method is that fictitious 'external' points need not be considered. This advantage is shared by the method presented in this paper.

The method can be extended in various ways. For example, it is being investigated how a similar kind of analysis could be established for slabs supported on beams.

If the loading consists of point loads, the results obtained should be treated with caution. The same can be said with regard to slabs supported by columns. It can be endeavoured to increase the accuracy by employing a finer network or by incorporating a more specifically analytic part in the solution procedure.

Finally, the author would like to express his indebtedness to various members of the Stevin Laboratory (Applied Mathematics section) and of the Institute T.N.O. for Building Materials and Building Structures for their valuable comments and suggestions.

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