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INVESTIGATION OF THE EFFECT OF THE BOUNDARY CONDITIONS ON THE LATERAL BUCKLING PHENOMENON, TAKING ACCOUNT OF CROSS SECTIONAL DEFORMATION

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INVESTIGATION OF THE EFFECT OF THE BOUNDARY CONDITIONS ON THE LATERAL BUCKLING PHENOMENON, TAKING ACCOUNT OF CROSS SECTIONAL DEFORMATION

Summary

The lateral buckling load of beams can be calculated by the energy method. With this method the effect of the deformation of the beam section on the lateral buckling load can be taken into account, which is not possible with the usual method of determining the lateral buckling load of beams.

Since the cross-sectional deformation for particular boundary conditions, depending on the length of the beam, may have a non-negligible influence on the lateral buckling load, the effect of this deformation on the lateral buckling load was investigated for various boundary conditions.

To this end, general expressions were established for the displacements of the beam sections and for the stresses in the beam, which, in combination with Bryan's energy equation, lead to a solution of the problem.

The types of support considered here are: forked bearings, semi-forked bearings and bearings by restraint of the bottom flange of the beam.

The results of the investigation were compared with those obtained with the usual method of analysis which is based on non-deformable sections and forked bearings and were verified with reference to a number of model tests.

Acknowledgments

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The English translation has been prepared by ir. C. van Amerongen.

Notation

A	cross-sectional area of a flange
A_1 to A_7	constants in the displacement function
a	web thickness
b	flange width
C_1 to C_8	constants in the displacement function
E	modulus of elasticity
e	flange thickness
G	shear modulus
h	depth of section
I_p	polar moment of inertia
I_t	torsional moment of inertia
I_z, I_y	moments of inertia about z-axis and y-axis
K	plate stiffness $= Ea^3/12(1-\mu^2)$
l	span of beam
M	support moment
$M(x)$	internal moment in beam
M_s	sum of support moments
P	load
Q	shear force
q	uniformly distributed load
$S(y)$	static moment
S_f	static moment of flange about neutral axis
T	work
U	strain energy
w	lateral displacement of beam
w_t	lateral displacement of top flange
w_b	lateral displacement of bottom flange
α	factor for unequal support moments $M_s = (1+\alpha)M$
β	restraint parameter $\beta = M_s/ql^2$
$\Delta\pi$	energy increase of beam
$\Delta\pi_b$	energy increase of bottom flange
$\Delta\pi_t$	energy increase of top flange
$\Delta\pi_w$	energy increase of web
μ	Poisson's ratio
σ_x, σ_y	normal stress in direction x and y
σ_{xy}	shear stress
σ_{xb}	normal stress in bottom flange
σ_{xt}	normal stress in top flange

Investigation of the effect of the boundary conditions on the lateral buckling phenomenon, taking account of cross sectional deformation

1 Introduction

The usual methods of analysis for the lateral buckling of beams (twist-bend buckling) are based on the assumption that, apart from warping, no deformations of the cross sections of a beam will occur [1]. Because of this assumption, only those types of bearing can be introduced into the analysis which, except for warping, do not allow any cross-sectional deformations to take place. The most familiar and, in terms of analytical technique, the simplest bearing of this kind is the forked bearing.

It has been established that, on this assumption, the calculated lateral buckling loads for rolled beam sections differ very little from the actual correct value. However, for compound sections, particularly those of large depth and with thin webs, this assumption results in too coarse an approximation. The lateral buckling load calculated in this way is liable to be much too large because, besides warping, considerable cross-sectional deformations may occur [2]. It will be investigated what effect such deformations have upon the lateral buckling load of rolled sections in the case where the bearings consist of forks, "semi"-forks or bottom flange restraints.

The standard of comparison adopted for this purpose is the lateral buckling load of a beam with non-deformable sections supported in forked bearings and analysed by the same method. Thus there are four different cases, which will now be considered in more detail:

Forked bearings and non-deformable beam sections (corresponding to the usual method of analysis):

By "non-deformable sections" is meant that the cross-section of the beam in the deflected position is unchanged (see Fig. 1.1). Warping of the section due to torsion and shear force may occur, however. The forked bearing prevents lateral displacement of the beam at the bearing and also prevents rotation of the beam about the longitudinal axis.

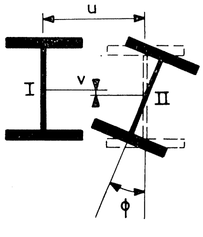


Fig. 1.1
Forked bearing, non-deformable beam sections.
I over a bearing II at mid-span

Rotations about axes perpendicular to the longitudinal axis of the beam, and translational displacement in the direction of the longitudinal axis, can freely occur.

Forked bearings and deformable beam sections

This second case differs from the preceding one only in that the deformation of the section in the span (the region between the supports) of the beam is now brought into the analysis. At the bearings the deformation of the beam section is prevented by the forked bearing arrangement (see Fig. 1.2).

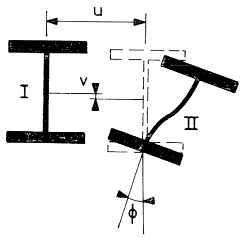


Fig. 1.2
Forked bearing, deformable beam sections.
I over a bearing II at mid-span

Semi-forked bearings and deformable sections

This third case differs from the second in that now a limited amount of deformation is possible also over the bearings. Since a semi-forked bearing prevents only the lateral displacements of the two flanges and prevents rotation of the bottom flange. Deformation of the beam web can occur at the bearings, together with simultaneous rotation of the top flange. In this case therefore all the sections of the beam can undergo deformation in the deflected position (see Fig. 1.3).

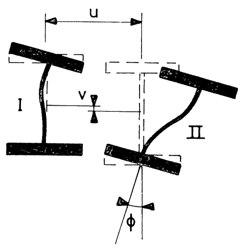


Fig. 1.3
Semi-forked bearing, deformable beam sections.
I over a bearing II at mid-span

Bearings by restraint of the bottom flange, deformable sections

In this last case the bearings prevent only the lateral displacement and the rotation of the bottom flange, whereas the top flange can undergo both lateral displacement and rotation at the bearings. So in this case, too, all the sections of the beam have undergone deformation in the deflected condition (see Fig. 1.4). Just as with the forked bearings, rotations about the axes perpendicular to the longitudinal axis of the beam, and translational displacement in the direction of that axis, are possible.

Warping of the end sections due to torsion and shear force is not restrained in any of the above-mentioned cases. Since the deformations cannot be brought into the analysis if the sections are considered as a whole, the beam is conceived as being composed of three parts. The flanges are regarded as members subject to flexural and torsional loading, while the web is regarded as a plate subject to double bending and torsion.

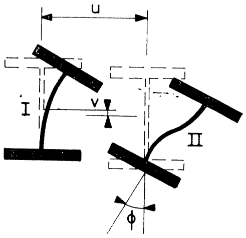


Fig. 1.4
Bottom flange restraint, deformable beam sections.
I over a bearing II at mid-span.

2 Principles of analysis

The loading case adopted is that of a uniformly distributed load acting on the top flange. This form of load application is a very unfavourable one with regard to lateral stability and is moreover of frequent actual occurrence.

The analysis is based on the following assumptions:

- The material is completely elastic, i.e., Hooke's law is unrestrictedly valid.
- The web of the beam section is sufficiently thin in relation to its depth to enable the theories of plates to be applied to it.
- The principal curvature of the beam at the instant of lateral buckling is small and therefore negligible.
- The beam has a thin-walled bisymmetrical I-shaped cross-section.
- The loading on the beam acts in the plane of the web and remains acting vertically when the beam has deflected laterally.
- There are no residual stresses in the beam section.
- The co-ordinate axes are assumed to be as indicated in Fig. 2.1.

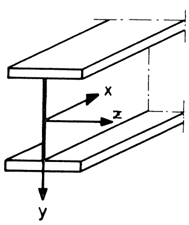


Fig. 2.1
System of co-ordinates adopted.

In general, the lateral buckling load of a beam is taken as equal to the smallest load for which, besides the undeformed state of equilibrium, a laterally deflected state of equilibrium is also possible. The beam is then in neutral equilibrium.

This lateral buckling load is calculated by means of the method of energy. To this end, an expression for the energy of the beam in an adjacent state of equilibrium is established. This energy can be expressed in displacements of, and stresses in, the beam section. The stresses can in turn be expressed in terms of the load, the dimensions of the beam, and the co-ordinates in the system of coordinate axes adopted.

For the displacements of the beam section a suitable method of expression, in the form of a trigonometric series, has been developed.

The constant coefficients in this series must be so determined that the load becomes a minimum. This minimum is the lateral buckling load which it is desired to determine.

3 Establishing the energy equation

Since the deformation of the web should be taken into account in the analysis, the beam is conceived as composed of three parts. The flanges are regarded as bar-type members loaded in bending and torsion, while the web is regarded as a plate subject to double bending and torsion. As a result of this subdivision of the section it is necessary to express the external loading of the beam in a system of internal stresses in the section.

The internal strain energy ΔU is provided by the stresses which are produced in the beam in consequence of the transition from the undeformed position to an adjacent deformed position of the beam.

The external work ΔT is provided by the stresses already present in the beam in consequence of the external loading at the instant of transition from the undeformed position to an adjacent deformed position.

Bryan's energy equation, which was derived for the instability of plates [3] can be applied to the web of the beam. This equation is as follows:

$$\Delta\pi_w = \tfrac{1}{2}\int_F \left\{ n_x \left(\frac{\partial w}{\partial x}\right)^2 + 2n_{xy}\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y} + n_y\left(\frac{\partial w}{\partial y}\right)^2 \right\} \mathrm{d}F +$$
$$+\frac{K}{2}\int_F \left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left\{\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right]\mathrm{d}F$$

where:

$\Delta\pi_w = \Delta U - \Delta T =$ increase in the energy of the web as a result of the deformation

$K = \dfrac{Ea^3}{12(1-\mu^2)} =$ plate stiffness of the web

$F = h \cdot l =$ area of the plate

l = length of the beam

h and a = depth and thickness of the web respectively

w = deflection of the plate in relation to the undeformed position

n_x, n_y, n_{xy} = normal forces in the plate.

On substitution of:

$$G = \frac{1}{2(1+\mu)}E$$

$$n_x = \sigma_x \cdot a;\ n_y = \sigma_y \cdot a;\ n_{xy} = \sigma_{xy} \cdot a$$

Bryan's equation becomes:

$$\Delta\pi_w = \frac{a}{2}\int_0^l \int_{-h/2}^{+h/2} (\sigma_x w_{,x}^2 + \sigma_y w_{,y}^2 + 2\sigma_{xy} w_{,x} w_{,y})\,\mathrm{d}x\,\mathrm{d}y +$$

$$+\frac{K}{2}\int_0^l \int_{-h/2}^{+h/2} (w_{,xx}^2 + w_{,yy}^2 + 2\mu w_{,xx} w_{,yy})\,\mathrm{d}x\,\mathrm{d}y + G\frac{a^3}{6}\int_0^l \int_{-h/2}^{+h/2} w_{,xy}^2\,\mathrm{d}x\,\mathrm{d}y \qquad (3.1)$$

The subscripts $,x$ and $,xy$ denote partial differences with respect to these quantities, so

$$w_{,x} \equiv \frac{\partial w}{\partial x} \quad \text{and} \quad w_{,xy} \equiv \frac{\partial^2 w}{\partial x \partial y}$$

For the flange the increase in energy on transition from the undeformed position to an adjacent deformed position we can make use of the general energy equation for bar-type members [4]:

$$\Delta\pi = \tfrac{1}{2}\int_0^l \left[EI_y u''^2 + E\Gamma\beta''^2 + GK\beta'^2 - Pu'^2 + 2Mu''\beta - P\left(\frac{I_p}{A} + \frac{eZ}{I_x}\right)\beta'^2 - \bar{a}W_y\beta^2 \right]\mathrm{d}z$$

where:

- $\Delta\pi$ = increase in energy $= \Delta U - \Delta T$
- P = an axial force in the member; in this equation, compression is reckoned as positive, whereas in the following calculations it is reckoned as negative
- W_y = distributed load perpendicular to the beam
- M = moment due to P, W_y and the eccentricity of P
- K = torsional moment of inertia, further designated as I_t
- I_p = polar moment of inertia
- A = cross-sectional area
- Γ = warping stiffness; for one flange we may put $\Gamma = 0$
- y_0 = distance from the line of action of P to the shear centre
- e = eccentricity of P in relation to the neutral axis
- u = horizontal deflection
- Z $= 2y_0 I_x - \int_A y(x^2+y^2)\,\mathrm{d}A$
- β = torsion angle
- I_y = moment of inertia in the weaker direction
- $\bar{a}$ = distance from load application point to shear centre

The co-ordinate axes xyz in this equation are replaced by the axes zyx.

If each of the flanges is considered separately as a "bar-type" member, then we can write by approximation for that member:

$M = 0$
$W_y = 0$ (the external load is replaced by an equivalent system of internal stresses; see Chapter 5)
$P/A = \sigma_x$ = normal stress in the flange
$\Gamma = 0$

Because of the symmetry of an I-section with respect to the x-axis in the system of co-ordinate axes xyz, in which the equation has been established, we have:

$$Z = 0$$

On substitution of the above the general energy equation for bar-type members becomes:

$$\Delta\pi = \tfrac{1}{2}\int_0^l \left[EI_y u''^2 + GI_t\beta'^2 + \sigma_x A u'^2 + \sigma_x I_p \beta'^2\right] \mathrm{d}x \tag{3.2}$$

Since the flange and the web are joined together, it is evident that the horizontal deflection "u" of the flange must be equal to the deflection "w" of the web at its junction with the flange. Also, the angular rotation β of the flange must be equal to the angular rotation of the deformed web at the web-to-flange junction. For the top flange these conditions of connection can be written as follows:

$$u \equiv w_t \quad \text{where} \quad w_t = w\left(y = -\frac{h}{2}\right)$$

$$\beta \equiv w_{t,y} \quad \text{where} \quad w_{t,y} = \frac{\partial w}{\partial y}\left(y = -\frac{h}{2}\right)$$

On substitution of these conditions into the general energy equation (3.2) we obtain for the energy increase $\Delta\pi_t$ of the top flange on transition from the undeformed position to the adjacent deformed position:

$$\Delta\pi_t = \tfrac{1}{2}\int_0^l \left[EI_y w_{t,xx}^2 + GI_t w_{t,yx}^2 + \sigma_{xt} A w_{t,x}^2 + \sigma_{xt} I_p w_{t,xy}^2\right] \mathrm{d}x \tag{3.3}$$

where:

I_y = moment of inertia about the y-axis
I_t = torsional moment of inertia
I_p = polar moment of inertia
A = cross-sectional area of flange
σ_{xt} = normal stress in top flange
w_t = displacement of top flange

The subscripts $,x$ and $,y$ denote partial differences with respect to these quantities. All the above-mentioned quantities refer to the top flange only.

In analogy with the above, the following equation for the energy increase $\Delta\pi_b$ can be established for the bottom flange:

$$\Delta\pi_b = \tfrac{1}{2}\int_0^l [EI_y w_{b,xx}^2 + GI_t w_{b,yx}^2 + \sigma_{xb} A w_{b,x}^2 + \sigma_{xb} I_p w_{b,xy}^2]\, dx \tag{3.4}$$

where:

σ_{xb} = normal stress in bottom flange
w_b = displacement of bottom flange

For the energy increase of the beam as a whole, on transition from the undeformed position to an adjacent deformed position, we now have:

$$\Delta\pi = \Delta\pi_w + \Delta\pi_t + \Delta\pi_b \tag{3.5}$$

This energy equation is generally-valid and can therefore be used for the analysis of the four cases mentioned in the introduction, provided that for each case a suitable mode of expression for the web deformation is employed.

4 Expressions for the deformations of the web

Forked bearings and non-deformable beam sections

Any arbitrary lateral displacement (to be referred to simply as "displacement" in the further treatment of the subject) of the web of a beam can be expressed as follows:

$$w(x, y) = w(x) \cdot w(y) \tag{4.1}$$

For beams with non-deformable sections the displacement at the instant of lateral buckling comprises a translation and a rotation (see Fig. 1.1). The following exact relationship can be given to present these displacements in the y-direction:

$$w(y) = A_1 + A_2 \cdot y$$

The bearings determine the course of the displacements in the x-direction. Since the forked bearing prevents rotation and translation relatively to the x-axis, all displacements of the web at such a bearing are zero. The course of the displacements in the x-direction is approximated by the first term of a trigonometric series:

$$w(x) = A_3 \sin \frac{\pi x}{l}$$

Substitution of the expressions for $w(y)$ and $w(x)$ into equation (4.1) yields the following expression for the displacement of the web:

$$w(x, y) = \sin\frac{\pi x}{l}(C_4+C_5\cdot y) \tag{4.2}$$

Forked bearings and deformable beam sections

For beams with deformable sections we can start from the generally-valid equation (4.1). From Fig. 1.2 it appears that in this case the course of the displacements of the web in the y-direction can be expressed as follows:

$$w(y) = w(y)_a + w(y)_b \tag{4.3}$$

where:

$w(y)_a$ = displacement of the undeformed sections due to rotation and translation

$w(y)_b$ = displacement due to deformation of the web

For $w(y)_a$ the same expression can therefore be used as in the case of non-deformable sections:

$$w(y)_a = A_1 + A_2\cdot y$$

The displacement $w(y)_b$ can be represented with fair approximation by the following expression (see Fig. 4.1):

$$w(y)_b = A_4\sin\frac{\pi y}{h} + A_5\cos\frac{\pi y}{h} + A_6\sin\frac{2\pi y}{h}$$

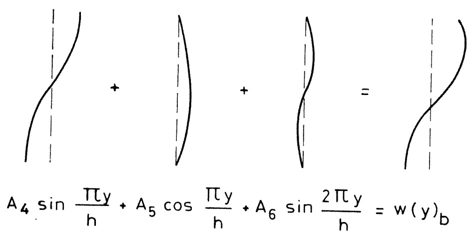


Fig. 4.1
Deformation of the web.

By a suitable choice of the constants A_4, A_5 and A_6 the deflection curve $w(y)_b$ can be approximated with sufficient accuracy.

Since we are in this case, too, considering forked bearings, we can adopt for $w(x)$ the same expression as in the preceding case:

$$w(x) = A_3\sin\frac{\pi x}{l}$$

On substituting the expressions for $w(x)$ and $w(y)$ into (4.1) we obtain, for this case, the following expression for the displacement of the web of the beam:

$$w(x, y) = \sin\frac{\pi x}{l}\left(C_4 + C_5 y + C_6 \sin\frac{\pi y}{h} + C_7 \cos\frac{\pi y}{h} + C_8 \sin\frac{2\pi y}{h}\right) \qquad (4.4)$$

Semi-forked bearings and deformable sections

In relation to the preceding case now only the boundary conditions differ, so that in this case the course of the web deformation in the y-direction can likewise be represented by the expression:

$$w(y) = w(y)_a + w(y)_b$$

and

$$w(y) = A_1 + A_2 \cdot y + A_4 \sin\frac{\pi y}{h} + A_5 \cos\frac{\pi y}{h} + A_6 \sin\frac{2\pi y}{h}$$

The course of the displacements in the x-direction, i.e., $w(x)$, is different from that in the preceding case. Since the origin of the co-ordinate axes is located on the neutral axis, however, it is merely necessary to add a constant term to the expression for $w(x)$, as obtained above, to give us the expression for the displacements in the x-direction for the present case:

$$w(x) = A_7 + A_3 \sin\frac{\pi x}{l}$$

As the end sections cannot undergo completely free displacements, a number of subsidiary conditions will also have to be satisfied. The bottom flange does not rotate, but is held in flat contact with the bearings by the pressure exerted by the load. Proceeding from the assumption that the web is rigidly connected to the flange, it follows that the tangent to the web at this rigid connection must be vertical. A further condition to be satisfied is that the top flange cannot undergo lateral displacement, which means that the web cannot undergo displacement at its junction with the top flange either. These conditions can also be expressed as follows:

$$w(x,y) = 0 \quad \text{for} \quad (x,y) = (0, +h/2) \quad \text{and} \quad (l, +h/2)$$

$$w_{,y}(x,y) = 0 \quad \text{for} \quad (x,y) = (0, +h/2) \quad \text{and} \quad (l, +h/2)$$

$$w(x,y) = 0 \quad \text{for} \quad (x,y) = (0, -h/2) \quad \text{and} \quad (l, -h/2)$$

Substitution of the expressions for $w(x)$ and $w(y)$, with these conditions, yields the following expression for the displacement of the web in this case:

$$w(x, y) = +\frac{\pi}{h}(C_2 + 2C_3)y - \frac{\pi}{2}(C_2 + 2C_3)\sin\frac{\pi y}{h} + C_2 \cos\frac{\pi y}{h} + C_3 \sin\frac{2\pi y}{h} + $$
$$+ \sin\frac{\pi x}{l}\left(C_4 + C_5 y + C_6 \sin\frac{\pi y}{h} + C_7 \cos\frac{\pi y}{h} + C_8 \sin\frac{2\pi y}{h}\right) \qquad (4.5)$$

Bearings by restraint of the bottom flange, deformable sections

The expressions for $w(x)$ and $w(y)$ for the case "semi-forked bearings and deformable sections" can be used in the present case also, except that in the subsidiary conditions the requirement as to non-displacement of the top flange now does not apply, so that only the following subsidiary conditions remain:

$$w(x,y) = 0 \quad \text{for} \quad (x,y) = (0, +h/2) \quad \text{and} \quad (l, +h/2)$$

$$w_{,y}(x,y) = 0 \quad \text{for} \quad (x,y) = (0, +h/2) \quad \text{and} \quad (l, +h/2)$$

In analogy with the preceding case we obtain the following expression for the displacement of the web of the beam:

$$w(x, y) = -\frac{\pi}{2}(C_2+2C_3)-C_1+\frac{\pi}{h}(C_2+2C_3)y+C_1\sin\frac{\pi y}{h}+ + C_2\cos\frac{\pi y}{h}+C_3\sin\frac{2\pi y}{h}+\sin\frac{\pi x}{l}\left(C_4+C_5 y+ + C_6\sin\frac{\pi y}{h}+C_7\cos\frac{\pi y}{h}+C_8\sin\frac{2\pi y}{h}\right) \tag{4.6}$$

By imposing certain restrictions as to this displacement it is possible to derive from this general expression for web displacement the expressions for the displacements in the three preceding cases.

With the extra condition $w(x, y) = 0$ for $(x, y) = (0, -h/2)$ and $(l, -h/2)$, which results in substitution of $C_1 = -\pi/2(C_2+2C_3)$, the expression (4.6) becomes the expression (4.5).

If the constant part (A_7) in the expression $w(x) = A_7+A_3 \sin \pi x/l$ is equated to zero, which means that $C_1 = C_2 = C_3 = 0$, the expression (4.6) becomes the expression (4.4).

If we additionally put $w(y)_b = 0$, hence $C_6 = C_7 = C_8 = 0$, we arrive at expression (4.2) again.

Summarising, it can be stated that from the general case of bearings by restraint of the bottom flange in conjunction with deformable sections it is possible to derive all the other cases by equating the appropriate coefficients to zero in the most comprehensive expression for the displacement of the web of the beam (4.6).

5 Expressions for the stresses

For determining the expressions for the stresses in the beam we shall start from the general loading case of a uniformly distributed load q acting on the top flange of the beam, with end moments M and αM over the supports, as indicated in Fig. 5.1. The

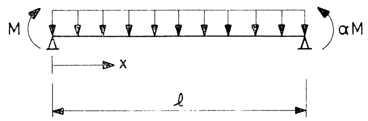


Fig. 5.1
Load arrangement considered.

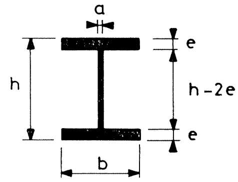


Fig. 5.2
Notation for I-section beam.

notation for the dimensions of the section is as shown in Fig. 5.2. The moment that occurs in the beam at a distance x from the support referred to is then:

$$M(x) = \tfrac{1}{2}qx(l-x) + M\left[1 - \frac{x}{l}(1-\alpha)\right] \tag{5.1}$$

The shear force Q as a function of x is given by the following expression:

$$Q(x) = \tfrac{1}{2}q(l-2x) - \frac{M}{l}(1-\alpha) \tag{5.2}$$

The normal stress σ_x in the beam thus becomes:

$$\sigma_x = \frac{M(x)\cdot y}{I_z} = \frac{y}{I_z}\left\{\tfrac{1}{2}qx(l-x) + M\left[1 - \frac{x}{l}(1-\alpha)\right]\right\} \tag{5.3}$$

In the top flange ($y = -h/2$) this normal stress will be:

$$\sigma_{xt} = \frac{-h}{2I_z}\left\{\tfrac{1}{2}qx(l-x) + M\left[1 - \frac{x}{l}(1-\alpha)\right]\right\} \tag{5.3a}$$

In the bottom flange ($y = +h/2$) this normal stress will be:

$$\sigma_{xb} = \frac{h}{2I_z}\left\{\tfrac{1}{2}qx(l-x) + M\left[1 - \frac{x}{l}(1-\alpha)\right]\right\} \tag{5.3b}$$

The shear stress $\sigma_{xy} = \sigma_{yx}$ in the web of the beam is:

$$\sigma_{xy} = \frac{Q(x)\cdot S(y)}{aI_z} = \frac{Q(x)}{aI_z}\left[S_f + \frac{a}{2}\left(\frac{h^2}{4} - y^2\right)\right]$$

Where S_f is the static moment of a flange with respect to the shear centre.
With equation (5.2) this equation becomes:

$$\sigma_{xy} = \frac{ql(l-2x)-2M(1-\alpha)}{2laI_z}\left[S - \frac{a}{2}y^2\right] \tag{5.4}$$

where

$$S = S_f + \frac{ah^2}{8}$$

The share of the flanges in the transmission of the shear force is therefore neglected.

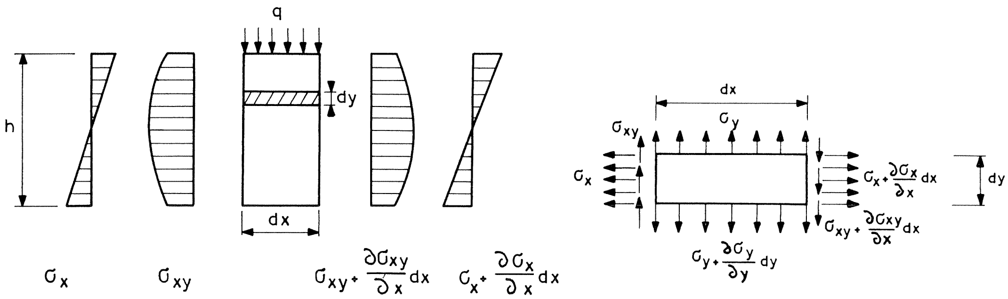


Fig. 5.3 Element of the beam Element of the web dx dy

From equilibrium of an element dx dy of the web of the beam the following expression is obtained for the normal stress σ_y (see Fig. 5.3):

$$\frac{\partial \sigma_y}{\partial y} \mathrm{d}y\, \mathrm{d}x + \frac{\partial \sigma_{xy}}{\partial x} \mathrm{d}x\, \mathrm{d}y = 0$$

$$\sigma_y = -\int \frac{\partial \sigma_{xy}}{\partial x} \mathrm{d}y = \frac{q}{aI_z}\left[Sy - \tfrac{1}{6}ay^3\right] + C$$

Basing ourselves on the assumption that the load q acts on the top flange of the beam, we obtain from this boundary condition (see Fig. 5.4):

$$y = 0{:}\ \sigma_y = -\frac{q}{2a} \quad \text{and therefore} \quad C = -\frac{q}{2a}$$

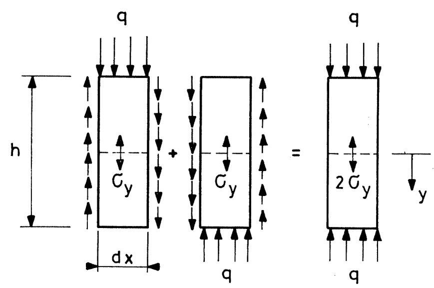


Fig. 5.4

Proof $\sigma_y = \frac{-q}{2a}$ for $y = 0$

The expression for the normal stress σ_y thus becomes:

$$\sigma_y = \frac{q}{aI_z}[Sy - \tfrac{1}{6}ay^3 - \tfrac{1}{2}I_z] \tag{5.5}$$

The expressions for the stresses derived above, together with expressions for the beam displacements derived in Chapter 4, can now be substituted into the energy equation.

6 Calculation of the lateral buckling load

The lateral buckling load is calculated by substituting the following functions into the energy equation (3.5)

$$\Delta\pi = \Delta\pi_w + \Delta\pi_t + \Delta\pi_b$$

- the displacements of the beam (equation 4.6) and the respective functions derived therefrom;
- the stresses (equations 5.3, 5.3a, 5.3b, 5.4 and 5.5).

As a result of this substitution we obtain energy equations for all four cases, the change in the energy quantity $\Delta\pi$ being expressed in:

- the section properties a, h, S, I_z, I_t, I_p, A:
  for a given section these quantities can be converted into numerical values;
- the material quantities E, G, μ:
  these can be converted into numerical values for a given material;
- the section property and material quantity K:
  for a given section and a given material this can be converted into a numerical value;
- the co-ordinates x, y:
  these are determined by the integration limits;
- the constants C_1 to C_8;
- the load q.

The constants C_1 to C_8 should now be so determined that the following two conditions are satisfied:

$\Delta\pi = 0$ (neutrality of the equilibrium);

$\Delta\pi =$ minimum (stable initial state):

$$\frac{\partial\,\Delta\pi}{\partial C_n} = 0$$

As a result of partial differentiation with respect to the constants C_1 to C_8 we obtain eight homogeneous linear equations in C_1 to C_8 from which the lateral buckling load q can be solved [5, 6].

This independent set of equations can be satisfied only in two ways, namely:

a. All the constants C_1 to C_8 are zero: this represents a trivial solution: the beam remains in its original position, i.e., q becomes zero.
b. The determinant of the equation written in the form of an 8×8 matrix must be zero. The values of the coefficients C_1 to C_8 then remain indeterminate.

From the matrix of coefficients for the case of restrained bottom flange and deformable sections the other three matrices of coefficients can be determined by means of supplementary conditions and by equating some of the constants to zero (analogy with the expressions for w: see Chapter 4). All the matrices of coefficients are symmetric with respect to the principal diagonal. The matrix of coefficients for the most general case of flange restraint (8×8 matrix) is given on page 27.

Here the matrix for the semi-forked bearing and deformable sections (7×7 matrix) can be derived by omission of the first row and first column, by adding to the second column $-\pi/2$ times the first column, and by adding to the third column $-\pi$ times the first column (corresponding to $C_1 = -\pi/2(C_2 + 2C_3)$: see Chapter 4).

The matrix for the forked bearing with deformable sections (5×5 matrix) is obtained by omission of the first three rows and columns (corresponding to $C_1 = C_2 = C_3 = 0$: see Chapter 4).

From this last-mentioned matrix the 2×2 matrix for the forked bearing with non-deformable sections is obtained by omission also of the last three rows and columns (corresponding to $C_6 = C_7 = C_8 = 0$: see Chapter 4).

From the matrix as a whole it appears that the effect of the support moments on the lateral buckling load is determined by the sum of the support moments $M_s = (1 + \alpha)M$.

7 Interpretation of the results of the analysis

A uniformly distributed load applied to the top flange is a very unfavourable method of loading on a beam with regard to the risk of lateral buckling; besides, it is of frequent occurrence. For this reason this type of loading has been chosen as the starting point for the present analysis. Other methods of loading can be considered by adapting the expressions for the stresses to such load types. With reference to a specific example (rolled steel section IPE 600: see Fig. 7.1) the lateral buckling load has been calculated with regard to the following four types of bearing, for the various values of the sum of the support moments $M(1+\alpha) = \beta q l^2$, with β as the restraint parameter:

- Forked bearings and non-deformable sections.
- Forked bearings and deformable sections.

- Semi-forked bearings and deformable sections.
- Bearings by bottom flange restraint, deformable sections.

It should once again be pointed out that this analysis is based on the following assumptions:

- Hooke's law is of unrestricted validity;
- there are no residual stresses in the section.

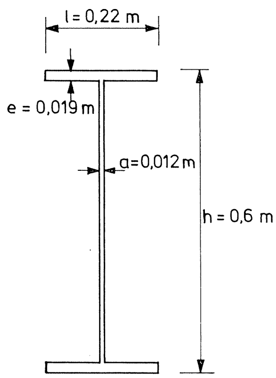


Fig. 7.1
Worked example IPE 600.

Forked bearings and non-deformable sections

This case is a first approximation of the ordinary case of lateral buckling. As can be expected, the results of this analysis are in good agreement with those obtained by the usual methods. Since the energy method constitutes the basis of this analysis and since we have contented ourselves with a first approximation of the deflection curve, the calculated values are somewhat too large (order of magnitude 1%).

Forked bearings and deformable sections (see Fig. 7.2)

The calculated lateral buckling loads for this case hardly differ from those obtained in the preceding case. It is evident that, even though the differences are small, the lateral buckling load in this case is certainly smaller than in the preceding one. This reduction of the lateral buckling load, caused by the deformation of the beam section, is not more than 4% in the region covered by the analysis. The greatest reduction will be found to occur in deep beams of short span. The said value of 4% relates to an IPE 600 section with a span of 6.00 m and restraint parameter $\beta = -0{,}15$. For an INP 600 section with the same span this reduction is less, this being due to the greater thickness of the web of this section. The effect of the deformation of the beam section is slight for beams consisting of IPE or INP sections, so that for rolled beams with forked bearings the assumption of non-deformability of the section is justified. For compound beams with very deep webs, on the other hand, such effect can indeed be considerable [2], so that in this latter case the cross-sectional deformation must be taken into account in the analysis.

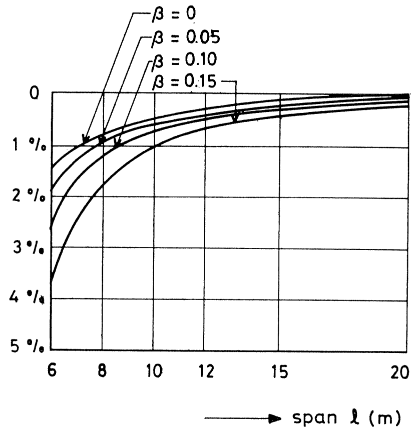


Fig. 7.2 Reduction of the lateral buckling load due to the effect of cross-sectional deformation in the mid-span region (forked bearings, deformable beam sections IPE 600).

Semi-forked bearings and deformable sections

It emerges that the restricted deformation of the beam section over the supports can considerably reduce the lateral buckling load. Even from this analysis it is evident how important the type of bearing of the beam may be with regard to lateral buckling stability.

Bearings by bottom flange restraint, deformable sections

The reduction of the lateral buckling load by the deformation of the beam section over the supports can be very considerable, especially for deep beams and short spans (see Fig. 7.3). From this diagram it appears that for short beams the lateral

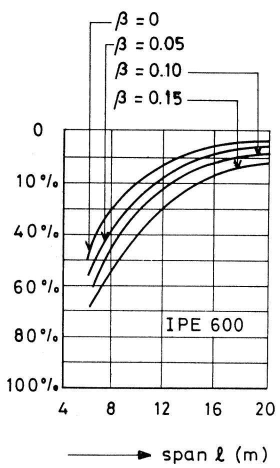


Fig. 7.3
Reduction of the lateral buckling load due to the effect of cross-sectional deformation in the mid-span region and over the supports, for IPE 600 (bottom flange restraint, deformable beam sections).

buckling load can decrease very greatly and that this decrease becomes more pronounced with decreasing values of the restraint parameter β, thus, for IPE 600 with a span $l = 8.00$ m and $\beta = -0.15$ it is more than 50%.

From the analysis of a large number of sections [5, 6] (not given here) it further appears that for sections having a thin deep web the lateral buckling load reduction is greater than for those with thicker or less deep webs.

In Fig. 7.4 the difference in lateral buckling load between the beam on forked bearings and the beam on bearings by bottom flange restraint is represented for a specific case, namely, IPE 600 and $\beta = 0$. From this diagram for the lateral buckling load it clearly emerges that the influence of bottom flange restraint is very great for small values of the length of the beam and that the influence of the type of bearing on the lateral buckling load diminishes with increasing length.

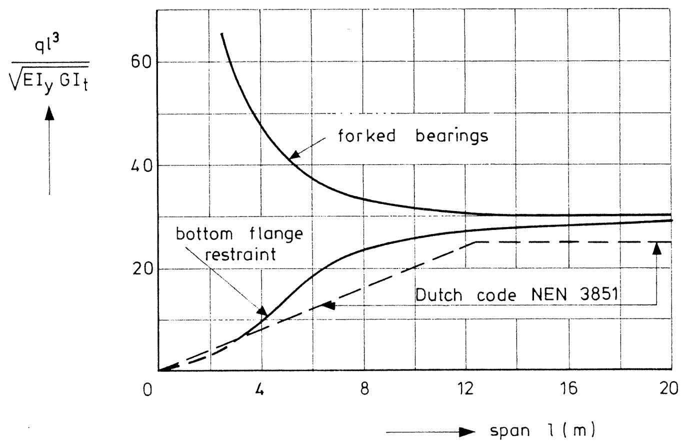


Fig. 7.4 Effect of the length of the beam on the lateral buckling load for IPE 600 beams with forked bearings and bottom flange restraint.

8 Experimental verification

Some model tests were carried out by the Institute TNO for Buildings Materials and Building Structures (TNO-IBBC) with a view to ascertaining whether the method of analysis for determining the lateral buckling load, as developed in Chapter 3, yields results that are in agreement with reality [7]. These tests were performed in two series. The first series comprised three tests on 1:5 scale model beams of an IPE 270 section (tests Nos. 1.1, 1.2 and 1.3); the second comprised eight tests on 1:10 scale model beams of an IPE 600 section (tests Nos. 2.11 to 2.38). All the model beams were loaded by uniformly distributed load on the top flange, simulated by point loads, which continued to act in the vertical direction during failure of the beams (gravity loading) (see Figs. 8.1 to 8.6).

Fig. 8.1
Test arrangement.

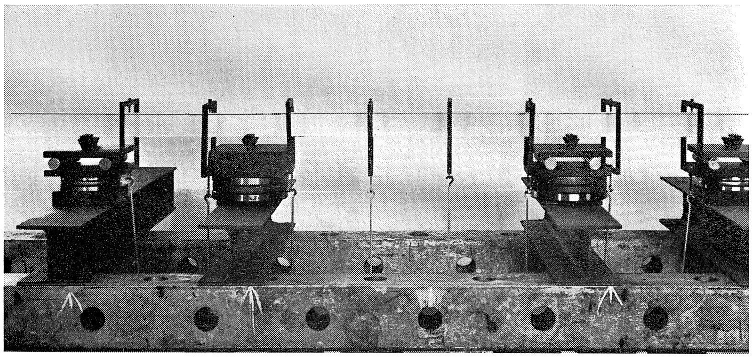


Fig. 8.2
Beam which has failed by lateral buckling over supports (test 2.11).

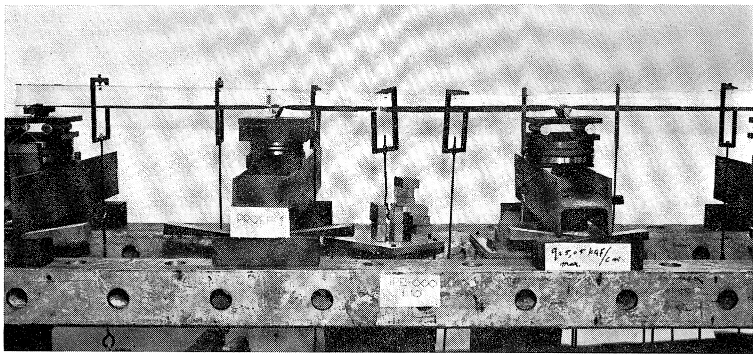
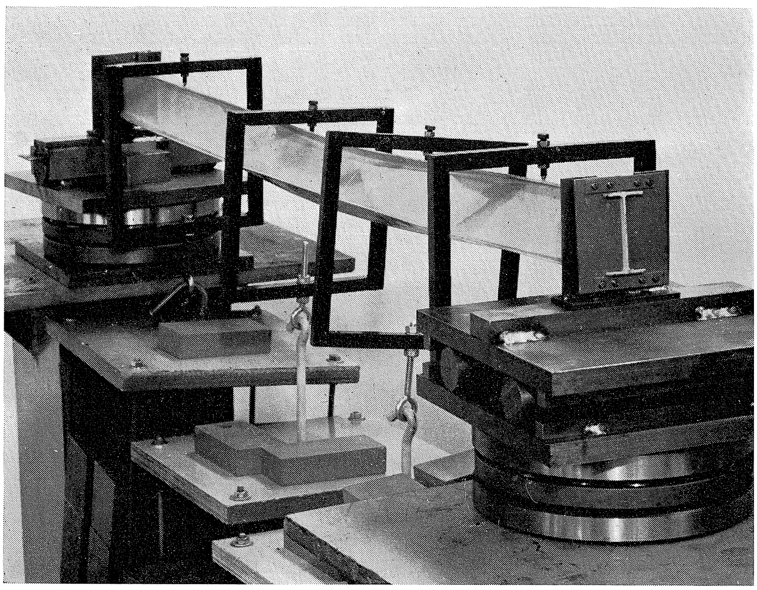
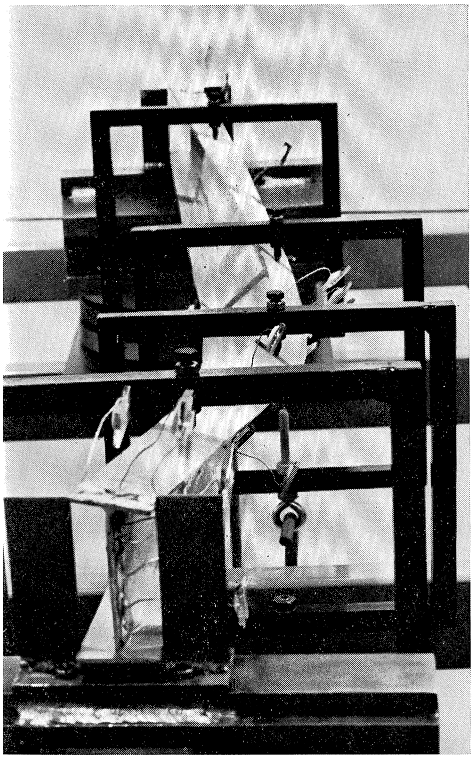


Fig. 8.3
Model test with forked bearings.





ig. 8.4 Semi-forked bearings (test 1.2).

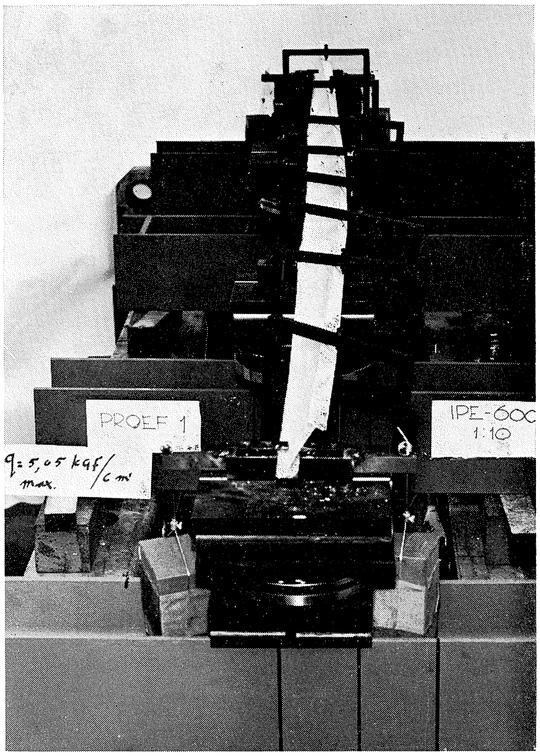
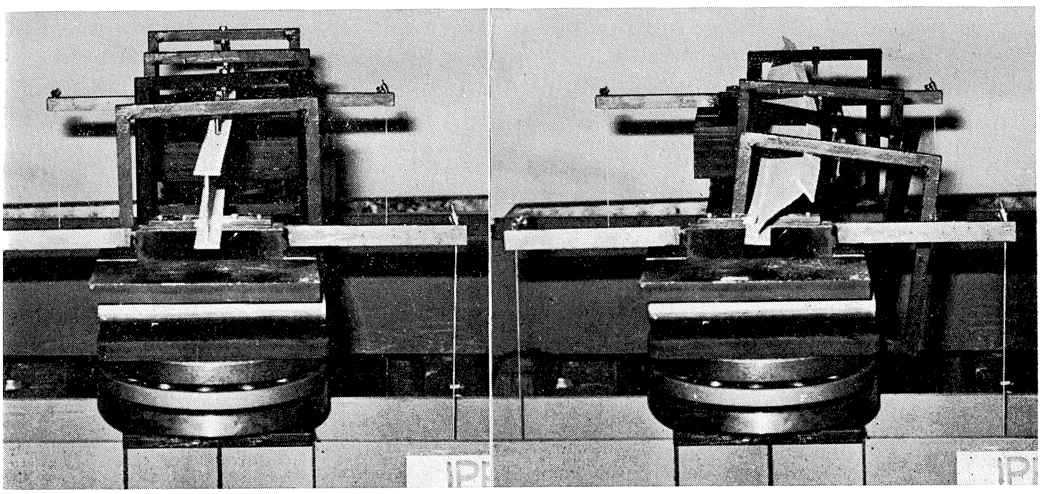


Fig. 8.6 Laterally buckled beam (test 2.11).

ig. 8.5 Beam just before and after failure (test 2.37).



The results of the tests are summarised in tabular form on page 25 and call for the following comments:

- The test results obtained with the beams continuous over several supports are all above the theoretical value. This must be attributed to the fact that the analysis neglects the lateral support which occurs over the bearings if the lateral buckling load of the adjacent span is larger than that of the span considered. The results will be better if all the spans of the beam become critical simultaneously.
- The lateral buckling load found in the tests does not fall short of the theoretical value by more than about 10%.
- The correctness of the method of analysis for bisymmetrical steel beams, as described in the present publication, is confirmed by the tests.

9 Concluding remarks

In the foregoing it has been shown that the effect of cross-sectional deformation of a beam on the lateral buckling stability can be calculated with the aid of the energy method. From a worked example relating to a rolled steel section (IPE 600) susceptible to lateral buckling (Fig. 7.1, 7.2, 7.3 and 7.4) it appears that only the deformations of the beam section over the supports significantly affect the lateral buckling stability of the beam. For long spans this effect diminishes, and for short spans it may become very great (see Fig. 7.4).

From a large number of calculations for rolled sections it emerges that cross-sectional deformation in the span of the beam always has little effect on lateral buckling stability and can therefore be neglected if forked bearings are used. The deformation of the beam section over the supports, on the other hand, has a considerable effect on the lateral buckling load in all cases with short spans and should then therefore be taken into account in the analysis.

The method of analysis presented here is, however, too complex for practical use and therefore difficult to apply. In order nevertheless to allow for the hazard of lateral buckling instability in short beams with bearings by bottom flange restraint, a simple rule of calculation [8] has been derived from a large number of calculations performed for rolled sections.

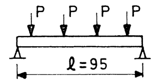
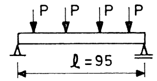
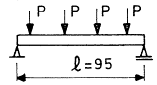
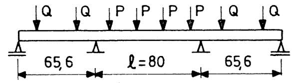
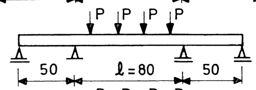
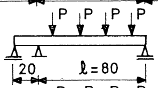
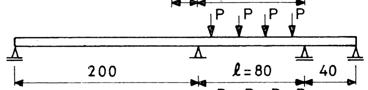
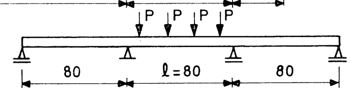
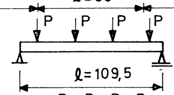
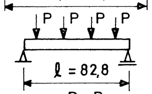
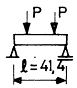
Since the results of these calculations are rather dependent on the particular section concerned, this simple rule constitutes a lower bound approximation which gives safe results in all cases and thus provides a conservative estimate of the critical lateral buckling load in many cases.

This rule of calculation has been included in the new Netherlands Building Code, Part: Steel Construction, NEN 3851, and is as follows:

If

$$l < 5\frac{be}{a}\sqrt{\frac{h}{a}}, \quad \text{then:} \quad q_{\text{lat.buckl.}} = 0.21(1+\beta)\frac{Ea^3}{lh}$$

In Fig. 7.4 this rule of calculation is indicated by a broken line.

test no.	test arrangements and load dispositions (dimensions in cm)		supports	$-M_L$	$-M_R$	$-M_s$	β	q^{*} test N/mm'	q_{theory} N/mm'	deviation from theory
1.1		$P=\frac{1}{4}q\ell$	forked bearings	o	o	o	o	4,6	4,75	+ 3,3%
1.2		$P=\frac{1}{4}q\ell$	semi-forked bearings	o	o	o	o	4,1	3,8	- 7,3%
1.3		$P=\frac{1}{4}q\ell$	bottom flange restraint	o	o	o	o	3,55	3,45	- 2,8%
2.11		$P=\frac{1}{4}q\ell$ $Q=\frac{1}{6}P$	"	$\frac{1}{17}q\ell^2$	$\frac{1}{17}q\ell^2$	$\frac{2}{17}q\ell^2$	0,1175	4,73	5,15	+ 7,8%
2.12		$P=\frac{1}{4}q\ell$	"	$\frac{1}{17}q\ell^2$	$\frac{1}{17}q\ell^2$	$\frac{2}{17}q\ell^2$	0,1175	4,73	5,05	+ 6,8%
2.23		$P=\frac{1}{4}q\ell$	"	$\frac{1}{10}q\ell^2$	o	$\frac{1}{10}q\ell^2$	0,1	4,55	4,15	- 8,8%
2.24		$P=\frac{1}{4}q\ell$	"	$\frac{1}{40}q\ell^2$	$\frac{3}{40}q\ell^2$	$\frac{1}{10}q\ell^2$	0,1	4,55	5,05	+ 11%
2.25		$P=\frac{1}{4}q\ell$	"	$\frac{1}{20}q\ell^2$	$\frac{1}{20}q\ell^2$	$\frac{1}{10}q\ell^2$	0,1	4,55	6,25	+ 37,4%
2.36		$P=\frac{1}{4}q\ell$	"	o	o	o	o	1.55	1,43	- 7,8%
2.37		$P=\frac{1}{4}q\ell$	"	o	o	o	o	3,31	3,1	- 6,3%
2,38		$P=\frac{1}{2}q\ell$	"	o	o	o	o	14,46	13,6	- 6,0%

* calculated with the correct dimensions of the model beams.

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