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LOWER BOUND APPROXIMATION FOR ELASTIC BUCKLING LOADS

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Notation

x, y	co-ordinates
l_i	length of member i
EI	flexural rigidity
P	numerical value for external force
λ	load factor
λ_e	load factor at which elastic buckling occurs
λ_u	upper bound for λ_e
λ_l	lower bound for λ_e
λ_u^*	upper bound for λ_u
$\lambda_1, \lambda_2 \ldots$	load factor first, second ... buckling mode
$u_y(x)$	displacement in y-direction
u_i	displacement at joint i
$M(x)$	bending moment
F_i	non-linear load at joint i
N_i	normal force in member i
θ_i	rotation of member i
P_{E_i}	"pin-ended Euler load" for member i
Δl_i	elongation of member i
E	elastic energy
W	work done by external forces
f	amplification factor

LOWER BOUND APPROXIMATION FOR ELASTIC BUCKLING LOADS

Summary

An approximate method for the elastic buckling analysis of two-dimensional frames is introduced. The method can conveniently be explained with reference to a physical interpretation: In the frame every member is replaced by two new members:

- a flexural member without extensional rigidity to transmit the shear force and the bending moments;
- a pin-ended rigid rocker member to transmit the normal force.

The buckling load of such a model can be calculated in a relatively simple manner. It is shown that, if no tensile forces occur in the frame, the buckling load of the model is an upper bound for the buckling load of the actual structure. By means of a simple formula a lower bound for the buckling load can then be determined. The method is more particularly of educational value. By means of this convenient and systematic method engineering students can fairly quickly gain an insight into the buckling behaviour of framed structures.

Lower bound approximation for elastic buckling loads

1. Introduction

The elastic buckling load may play an important part in the assessment of the load-carrying capacity of a structure.

A good illustration of this is provided by the Rankine type formula proposed by Merchant (ref. [1], [2], [4]). In this formula the maximum force that the structure can support is estimated from the elementary collapse load and the elastic buckling load:

$$\frac{1}{\lambda_c} = \frac{1}{\lambda_p} + \frac{1}{\lambda_e}$$

where:

λ_p = load factor for which the structure collapses according to a geometrically linear analysis;
λ_e = load factor for which the structure buckles elastically;
λ_c = load factor for which the maximum carrying capacity of the structure is reached under the influence of plastification and second-order effects.

The accuracy of this estimate for λ_c will of course depend to a great extent on circumstances. For obtaining an approximate preliminary indication the formula is very suitable, however.

A mathematically exact determination of the buckling load by solving differential equations soon becomes very complicated if a structure comprising more than two or three members is dealt with. Over the years a considerable number of approximate methods have therefore been developed [3], [5], [6], [7], among which the computer-oriented methods nowadays predominate [8]. Cases may arise, however, in which it is undesirable (e.g., for educational reasons) to perform a complete computer analysis. The need for a simple and systematic approximate method then exists [9].

In this publication such a method will be explained with reference to a number of examples (Part I). The origin of the method is based on [7]. In the first instance upper bound solutions are obtained. By a simple procedure, however, lower bounds can also be established. It is important to remark that for practical cases it is not necessary to calculate λ_e with great accuracy. In general λ_e is 3 à 10 times larger in magnitude than λ_p. So a small deviation (10 à 20%) in λ_e has only a minor influence on λ_c.

Finally, in Part II the proof for the validity of the method is presented.

2. Fixed-end member, one degree of freedom

A member with length l and flexural rigidity EI, fixed (rigidly gripped) at the base, is loaded in compression by a vertical force λP (fig. 1). P represents a given load, e.g., corresponding to the estimated working load; λ is a load factor. Find the value of the load factor $\lambda = \lambda_e$ for which this strut buckles elastically.

The equilibrium method will be used to solve this problem, i.e., a load factor λ_e and a deflected shape $u_y(x) \neq 0$ will be sought for which the strut is in a state of equilibrium.

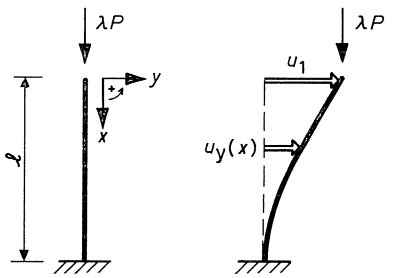

Fig. 1. Fixed-end member (strut).

Exact solution

The usual exact procedure for solving the problem is as follows [5]:

The deflection $u_y(x)$ and the load λP will cause second-order moments to develop in the strut, namely:

$$M(x) = \lambda P\{u_1 - u_y(x)\} \tag{1}$$

where $u_1 = u_y(x=0)$. The value of the internal bending moments associated with the deflection $u_y(x)$ is determined through the moment-curvature relation:

$$M(x) = EIu_{y,xx}(x) \tag{2}$$

Combination of (1) and (2) gives a differential equation for u_y:

$$EIu_{y,xx} + \lambda Pu_y = \lambda Pu_1 \tag{3}$$

Together with the boundary conditions $u_y(l) = u_{y,x}(l) = 0$, and $u_y(0) = u_1$, this equation yields, by the familiar procedure, the following expressions for the buckling load and the buckled shape (buckling curve):

$$\lambda_e = \frac{\pi^2 EI}{4Pl^2} = 2.47\frac{EI}{Pl^2} \tag{4}$$

$$u_y(x) = u_1\left\{1 - \sin\left(\frac{\pi x}{2l}\right)\right\} \qquad \text{with } u_1 = \text{indeterminate}$$

Approximation

Many approximate methods for dealing with buckling problems are based on the advance estimation of $u_y(x)$ in formula (1) [5], [6], [7]. The estimate may comprise one or more parameters. Next, a function $u_y(x)$ can be calculated with the aid of the differential equation. By comparing the estimated u_y with calculated u_y a buckling load can be determined. The accuracy of this approximate method depends on:

- the extent to which the estimated function u_y satisfies the boundary conditions;
- the number of parameters or degrees of freedom that is introduced;
- the manner in which the estimated and the calculated deflection functions u_y are compared with each other.

The approximation in the present paper will be based on a linear estimation formula with one parameter:

$$u_y(x) = u_1 \left\{ 1 - \frac{x}{l} \right\} \tag{5}$$

The boundary condition $u_{y,x}(l) = 0$ is therefore not satisfied. In conjunction with (1) and (2) expression (5) leads to the differential equation:

$$EIu_{y,xx} = \lambda P \frac{u_1}{l} x$$

On twice integrating and making use of the boundary conditions $u_y(l) = u_{y,x}(l) = 0$ we obtain:

$$u_y = \frac{\lambda P u_1}{EIl} \{\tfrac{1}{6}x^3 - \tfrac{1}{2}xl^2 + \tfrac{1}{3}l^3\} \tag{6}$$

Equating the two functions u_y given by (5) and (6) at $x = 0$ yields:

$$u_1 = \frac{\lambda P l^2 u_1}{3EI}$$

Solutions with $u_1 \neq 0$ are possible only if:

$$\lambda = \lambda_2 = \frac{3EI}{Pl^2} \tag{7}$$

The approximation gives an over-estimation of 20% with respect to the exact solution (4).

Physical model of the approximation

The approximation method described in the foregoing can be interpreted quite simply in physical terms [7]. Suppose the member under consideration (vertically loaded strut) to be split up into:

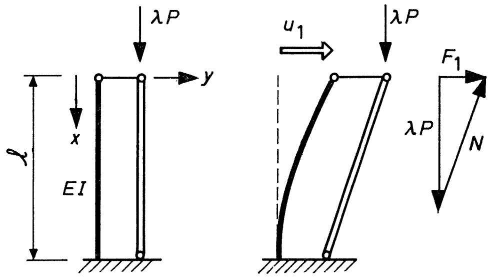

Fig. 2. Physical model of the approximation.

- a rigid rocker member (i.e., pin-jointed at both ends) which has to transmit the normal (direct) force;
- a flexural member, without extensional rigidity, which resists the bending moments and shear forces.

The actual structure is thus represented by a model as shown in fig. 2. For convenience of presentation the rocker member and the flexural member are shown side by side.

The exact buckling analysis for the model substituted for the actual structure proceeds as follows:

In the inclined position the normal force N acting in the rocker member and the vertical load are not in equilibrium with each other. As a result, a horizontal force of the following magnitude acts at the top of the flexural member:

$$F_1 = \lambda P \frac{u_1}{l} \tag{8}$$

This horizontal force F_1 can be conceived as a "non-linear load" in analogy with d'Alembert's "inertia load". In consequence of the load F_1 the top of the flexural member undergoes a horizontal displacement:

$$u_y(0) = \frac{F_1 l^3}{3EI} \tag{9}$$

On substituting (8) into (9) and equating $u_y(0)$ and u_1 we obtain:

$$u_1 = \frac{\lambda P l^2}{3EI} u_1$$

whence we find the critical load factor:

$$\lambda_e = \frac{3EI}{Pl^2} \tag{7}$$

It can readily be seen that the model is entirely in agreement with the approximate method described earlier on. The rocker member performs the role of the estimated linear deflection function u_y as expressed by (5).

In the further treatment of the subject the model will always be adopted as the basis. The advantages are that extension of the procedure to more complex structures is simpler and that also the application of energy methods (Part II) is directly possible.

3. Fixed-end member, two degrees of freedom

An improvement in accuracy is to be expected if the member is subdivided into several elements. Here the case where it is subdivided into two elements will be considered. The physical model is shown in fig. 3.

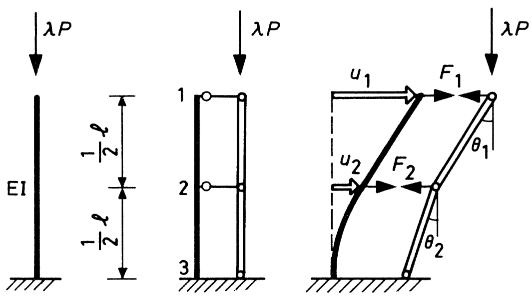

Fig. 3. Fixed-end member, model with two degrees of freedom.

As a result of the displacements u_1 and u_2 of the joints 1 and 2 the forces F_1 and F_2 are exerted on the flexural member:

$$F_1 = \lambda P \sin\theta_1$$
$$F_2 = \lambda P(\sin\theta_2 - \sin\theta_1)$$

θ_1 and θ_2 are the angles between the rocker members and the vertical. Expressed in terms of u_1 and u_2:

$$\begin{aligned} F_1 &= \lambda P\left[\frac{u_1-u_2}{\frac{1}{2}l}\right] = 2\lambda P(u_1-u_2)/l \\ F_2 &= \lambda P\left[\frac{u_2}{\frac{1}{2}l} - \frac{u_1-u_2}{\frac{1}{2}l}\right] = 2\lambda P(2u_2-u_1)/l \end{aligned} \qquad (10)$$

The displacements u_1 and u_2 of the flexural member in consequence of the non-linear load are, according to the linear theory of elasticity:

$$\begin{aligned} u_1 &= \frac{F_1 l^3}{3EI} + \frac{5F_2 l^3}{48EI} \\ u_2 &= \frac{5F_1 l^3}{48EI} + \frac{F_2 l^3}{24EI} \end{aligned} \qquad (11)$$

On substitution of (10) into (11):

$$u_1 = \frac{2\lambda P(u_1 - u_2)l^2}{3EI} + \frac{10\lambda P(2u_2 - u_1)l^2}{48EI}$$

$$u_2 = \frac{10\lambda P(u_1 - u_2)l^2}{48EI} + \frac{2\lambda P(2u_2 - u_1)l^2}{24EI}$$

Rearrangement:

$$\left\{1 - 11\frac{\lambda Pl^2}{24EI}\right\} u_1 + \left\{\quad 6\frac{\lambda Pl^2}{24EI}\right\} u_2 = 0$$

$$\left\{\quad - \ 3\frac{\lambda Pl^2}{24EI}\right\} u_1 + \left\{1 + \frac{\lambda Pl^2}{24EI}\right\} u_2 = 0$$

This set of linear homogeneous equations in u_1 and u_2 has non-zero solutions only if the determinant of the set is zero:

$$\left\{1 - 11\frac{\lambda Pl^2}{24EI}\right\}\left\{1 + \frac{\lambda Pl^2}{24EI}\right\} - \left\{6\frac{\lambda Pl^2}{24EI}\right\}\left\{-3\frac{\lambda Pl^2}{24EI}\right\} = 0$$

The solutions of this second-degree equation are:

$$\lambda_1 = 2.59\frac{EI}{Pl^2} \quad \text{and} \quad \lambda_2 = 31.6\frac{EI}{Pl^2} \tag{12}$$

The lower of these two values is the desired approximation for λ_e. The error is now 4%.

The associated buckled shape is given by:

$$\frac{u_1}{u_2} = 3.40$$

For the exact solution the corresponding value is:

$$\frac{u_1}{u_2} = 3.41$$

It can be concluded that the buckling problem of the fixed-end member can be solved with fair accuracy even with just two elements.

4. Simple frame

The frame shown in fig. 4a (all its members have flexural rigidity EI) is loaded at joint 3 by a vertical force λP. Determine the elastic buckling load with the aid of the approximation model.

In fig. 4b the frame is shown in a deflected position. The horizontal displacement of the top member is $u_3(= u_2)$. The rocker members in the physical model are represented by dotted lines.

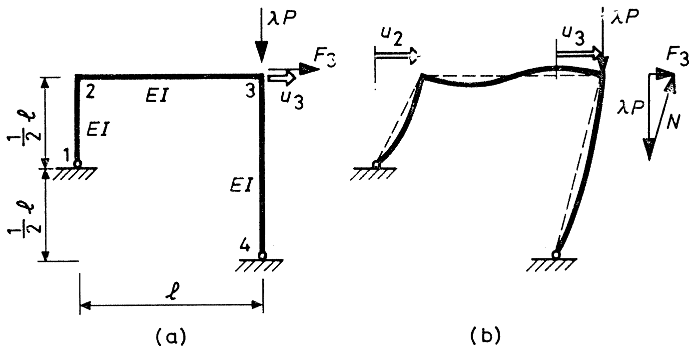

Fig. 4 Frame with one degree of freedom.

In the vertical (undeflected) position a normal force $N = -\lambda P$ is acting in the rocker member 3–4; all other internal forces are zero. In the deflected position the normal force in the rocker member 3–4 is no longer in equilibrium with the load. As a result, a non-linear load acts on the frame, namely, a horizontal force F_3 acting at joint 3:

$$F_3 = \lambda P \frac{u_3}{l} \tag{13}$$

With the aid of the linear elastic theory it can readily be shown that the displacement u_3 caused by F_3 is expressed by:

$$u_3 = 0.080 \frac{F_3 l^3}{EI} \tag{14}$$

Substitution of (13) into (14) gives:

$$u_3 = 0.080 \frac{\lambda P l^2}{EI} u_3$$

From this we obtain as the approximation for λ_e:

$$\lambda_e = 12.5 \frac{EI}{Pl^2} \tag{15}$$

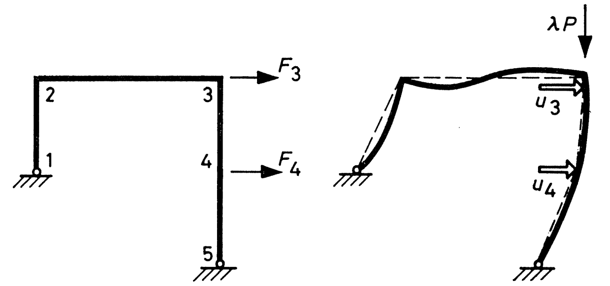

Fig. 5. Frame with two degrees of freedom.

This result can probably be improved by subdividing the long vertical member of the frame into two elements. In that case an analysis involving two degrees of freedom must be performed (see fig. 5).

The non-linear load due to the inclined (deflected) position of the frame is, in analogy with the formula (10) for the problem of the fixed-end member, expressed by:

$$\begin{aligned} F_3 &= 2\lambda P\{\ u_3 - u_4\}/l \\ F_4 &= 2\lambda P\{2u_4 - u_3\}/l \end{aligned} \tag{16}$$

Analysis in accordance with the linear elastic theory gives:

$$\begin{aligned} u_3 &= \{0.080F_3 + 0.054F_4\}l^3/EI \\ u_4 &= \{0.054F_3 + 0.050F_4\}l^3/EI \end{aligned} \tag{17}$$

On substitution of (16) into (17):

$$u_3 = \{0.080(2u_3 - 2u_4) + 0.054(4u_4 - 2u_3)\}\lambda Pl^2/EI$$

$$u_4 = \{0.054(2u_3 - 2u_4) + 0.050(4u_4 - 2u_3)\}\lambda Pl^2/EI$$

Rearrangement again yields two homogeneous linear equations in u_3 and u_4. On equating the determinant to zero we obtain:

$$\lambda_1 = 9.9\frac{EI}{Pl^2} \quad \text{and} \quad \lambda_2 = 22.2\frac{EI}{Pl^2} \tag{18}$$

The lower value is the desired approximation for λ_e. The buckled shape is given by (see fig. 6):

$$\frac{u_3}{u_4} = 1.1$$

The model with one degree of freedom gives as the approximation for the buckling load:

$$\lambda = 12.5\frac{EI}{Pl^2}$$

The exact analysis gives:

$$\lambda_e = 8.8\frac{EI}{Pl^2}$$

The relative errors are therefore:

- model with one degree of freedom: +42%
- model with two degrees of freedom: +11%.

To clarify the difference between the two answers, let us have another look at fig. 6.

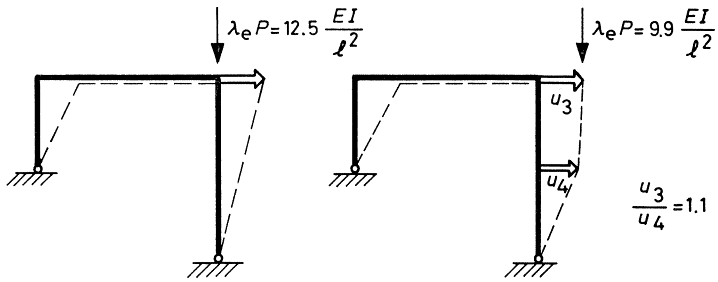

Fig. 6. Buckling shapes of both models.

In the first model (one degree of freedom) the rocker member of the right column prescribes the value of u_3/u_4 at 2.

In the second model this value is 1.1. This gives rise to the conclusion that only a good approximation can be obtained when the rocker members do permit a shape which is fairly close to the real buckling model.

5. Lower bound approximation

The approximations so far presented have in all cases over-estimated the actual buckling load. This is obvious, because the rocker members can be regarded as extra stiffening members associated with the structure. A formal proof of this is presented in Part II. It emerges that the condition for the existence of the upper bound is that there must be no tensile forces in the structure. Subject to the same restriction it proves possible also to establish a lower bound for the buckling load. The actual buckling load can therefore be enclosed between two values. This clearly enhances the value of the approximate method.

The formula for the lower bound is:

$$\frac{1}{\lambda_l} = \frac{1}{\lambda_u} + \left[\frac{-N_i}{P_{E_i}}\right]_{\max} \tag{19}$$

where:

λ_u = upper bound associated with a particular model
λ_l = lower bound associated with the same model
N_i = normal force in member i for $\lambda = 1$ (tension is positive)

$$P_{E_i} = \frac{\pi^2 EI_i}{l_i^2}$$

where:

EI_i = flexural rigidity of element i
l_i = length of element i

The proof of the lower bound formula will be given in Part II. At this moment we will suffice to make the formula plausible.

First of all there is a need for a sharp definition of the normal force. In the following the normal force will be understood as a force transmitted by the frame-member and whose line of action passes through the member end joints. The definition holds in the undeformed state as well as in the deformed state.

In the model the normal force is transmitted by the rocker member. In reality of course, this rocker member does not exist and the frame member itself has to transmit the normal force. From this some non-linear effects result, which have not been taken into account yet.

It is a well known phenomena that the bending rigidity of a frame member is reduced if normal pressure forces are present.

Consider the beam of fig. 7. According to the elementary second order theory the member end rotation can be calculated from:

$$\varphi = \frac{M}{2EI} \qquad \text{(no normal force present)}$$

$$\varphi \simeq f \cdot \frac{M}{2EI} \qquad \text{(at presence of normal force } N\text{)}$$

where

$$f = \frac{P_E}{P_E + N} = \text{so called amplification factor}$$

$$P_E = \frac{\pi^2 EI}{l^2} = \text{elementary Euler load}$$

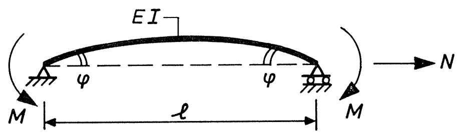

Fig. 7. Beam loaded by equal end moments and normal force.

From the above expressions it can be concluded that approximately the bending stiffness EI is reduced to EI/f. This provides a very simple way of taking into account the non-linear effects produced by N: we just divide the member stiffness by f. There is only one problem: in the actual structure the ratio of the end moments may differ from 1; this ratio may become 0 or even -1. Especially when the ratio is close to -1 the proposed approximation will not be very accurate. However, in general the approximation underestimates the stiffness properties of the member. So in looking for a lower bound it still will be usable.

To simplify matters further we should like to have one and the same reduction factor for all members. This reduction factor of course has to be the largest number

f present; let us call this number f_{max}. All member stiffnesses being reduced in the same proportion, we can calculate the lower bound directly from the upper bound through:

$$\lambda_l = \frac{\lambda_u}{f_{max}}$$

Substituting:

$$f_{max} = \left[\frac{P_E}{P_E + N}\right]_{max}$$

$$\lambda_l = \frac{\lambda_u(P_E + N)}{P_E} = \lambda_u + \frac{\lambda_u N}{P_E}$$

The reference for the heaviest loaded member is omitted here.

Dividing by $\lambda_l \cdot \lambda_u$ leads to:

$$\frac{1}{\lambda_u} = \frac{1}{\lambda_l} + \frac{N}{P_E}$$

From this it is a small step to (19).

We will proceed now by applying the lower bound formula to the foregoing problems:

Fixed-end member, one degree of freedom (fig. 2):

Calculated value for the upper bound (formula (7)):

$$\lambda_u = 3\frac{EI}{Pl^2}$$

There is only one element, with:

$$N_1 = -P$$

$$P_{E_1} = \frac{\pi^2 EI}{l^2}$$

Employing the lower bound formula (19):

$$\frac{1}{\lambda_l} = \frac{1}{3}\left[\frac{Pl^2}{EI}\right] + \frac{1}{\pi^2}\left[\frac{Pl^2}{EI}\right]$$

$$\lambda_l = 2.31\frac{EI}{Pl^2}$$

Fixed-end member, two degrees of freedom (fig. 3, formula (12)):

$$\lambda_u = 2.59\frac{EI}{Pl^2}$$

For the two elements:

$$N_i = -P$$

$$P_{E_i} = \pi^2 EI/(\tfrac{1}{2}l)^2$$

The lower bound:

$$\frac{1}{\lambda_l} = \frac{1}{2.59}\left[\frac{Pl^2}{EI}\right] + \frac{1}{4\pi^2}\left[\frac{Pl^2}{EI}\right]$$

$$\lambda_l = 2.43\frac{EI}{Pl^2}$$

Portal frame, one degree of freedom (fig. 4):

Upper bound according to (15)

$$\lambda_u = 12.5\frac{EI}{Pl^2}$$

The structure is subdivided into three elements. In the short vertical member and in the horizontal member the normal force is zero. For the lower bound formula the long vertical member is therefore determinative, with:

$$N = -P$$

$$P_E = \pi^2 EI/l^2$$

Employing the lower bound formula:

$$\frac{1}{\lambda_l} = \frac{1}{12.5}\left[\frac{Pl^2}{EI}\right] + \frac{1}{\pi^2}\left[\frac{Pl^2}{EI}\right]$$

$$\lambda_l = 5.5\frac{EI}{Pl^2}$$

The great difference between the upper and the lower bound is an indication that the buckling behaviour is very imperfectly described by an approximation model comprising only one degree of freedom.

Portal frame, two degrees of freedom (fig. 5, formula (18)):

$$\lambda_u = 9.9\frac{EI}{Pl^2}$$

For the two elements of the long vertical member:

$$N = -P \quad \text{and} \quad P_E = \pi^2 EI/(\tfrac{1}{2}l)^2$$

so that:

$$\frac{1}{\lambda_l} = \frac{1}{9.9}\left[\frac{Pl^2}{EI}\right] + \frac{1}{4\pi^2}\left[\frac{Pl^2}{EI}\right]$$

$$\lambda_l = 7.9\frac{EI}{Pl^2}$$

For this approximation model the lower and the upper bound are much closer together.

Summary of the results obtained

(λ_e = exact value)

structure	degrees of freedom	λ_u/λ_e	λ_l/λ_e
fixed-end member	1	1.21	0.93
fixed-end member	2	1.04	0.98
portal frame	1	1.42	0.63
portal frame	2	1.12	0.90

Finally, a few examples of somewhat larger structures will be presented in the next paragraph.

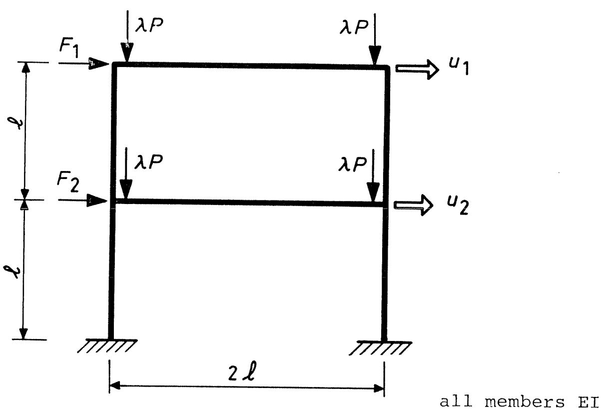

Fig. 8. Two-storey frame.

6. Examples

Two-storey frame

For the framed structure shown in fig. 8 determine the upper and the lower bound for the critical load factor.

Two degrees of freedom u_1 and u_2 are assigned to the structure. The non-linear load according to the rocker member model is:

$$F_1 = 2\lambda P(u_1 - u_2)/l \qquad = \lambda P(2u_1 - 2u_2)/l$$

$$F_2 = 4\lambda P u_2/l - 2\lambda P(u_1 - u_2)/l = \lambda P(6u_2 - 2u_1)/l$$

Linear elastic calculations yield the following result:

$$u_1 = (0.25F_1 + 0.10F_2)l^3/EI$$

$$u_2 = (0.10F_1 + 0.07F_2)l^3/EI$$

Elimination of F_1 and F_2 gives two homogeneous linear equations in u_1 and u_2. On equating the determinant to zero, two roots are obtained:

$$\lambda_1 = 2.9\frac{EI}{Pl^2} \quad \text{and} \quad \lambda_2 = 5.7\frac{EI}{Pl^2}$$

The lower value λ_1 is the desired upper bound approximation for λ_e.

The associated buckled shape is characterized by:

$$\frac{u_1}{u_2} = 2.1$$

The lower bound is determined by the two columns of the bottom storey, with:

$$N = -2P \quad \text{and} \quad P_E = \pi^2 EI/l^2$$

so that:

$$\frac{1}{\lambda_l} = \frac{1}{2.9}\left[\frac{Pl^2}{EI}\right] + \frac{2}{\pi^2}\left[\frac{Pl^2}{EI}\right]$$

$$\lambda_l = 1.9\frac{EI}{Pl^2}$$

The difference between the upper and the lower bound is arithmetically determined entirely by the bottom storey columns. To improve the accuracy of the results the obvious thing to do is to subdivide the two bottom storey columns each into two elements.

Three-storey frame

For this framed structure a model comprising three degrees of freedom (u_1, u_2, u_3) will be adopted.

The non-linear load according to the rocker member model is:

$$F_1 = 2\lambda P(u_1 - u_2)/l \qquad = \lambda P(\;\;2u_1 - 2u_2 \qquad)/l$$

$$F_2 = 4\lambda P(u_2 - u_3)/l - 2\lambda P(u_1 - u_2)/l = \lambda P(-2u_1 + 6u_2 - \;4u_3)/l$$

$$F_3 = 6\lambda P(u_3 \qquad)/l - 4\lambda P(u_2 - u_3)/l = \lambda P(\qquad -4u_2 + 10u_3)/l$$

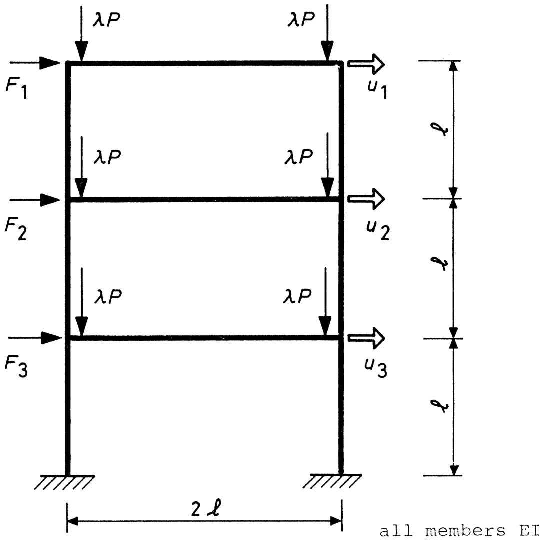

Fig. 9. Three-storey frame.

With the aid of the linear elastic theory we obtain:

$$u_1 = (0.452F_1 + 0.287F_2 + 0.107F_3)l^3/EI$$

$$u_2 = (0.287F_1 + 0.238F_2 + 0.099F_3)l^3/EI$$

$$u_3 = (0.107F_1 + 0.099F_2 + 0.068F_3)l^3/EI$$

By applying the usual procedure we arrive at the upper bound approximation:

$$\lambda_u = 1.65\frac{EI}{Pl^2}$$

For the lower bound the columns of the bottom storey are again the determining members, with:

$$N = -3P \quad \text{and} \quad P_E = \pi^2 EI/l^2$$

so that:

$$\frac{1}{\lambda_l} = \frac{1}{1.65}\left[\frac{Pl^2}{EI}\right] + \frac{3}{\pi^2}\left[\frac{Pl^2}{EI}\right]$$

$$\lambda_l = 1.10\frac{EI}{Pl^2}$$

Arch structure

Determine the critical load factor λ_e for the schematized arch structure shown in fig. 10.

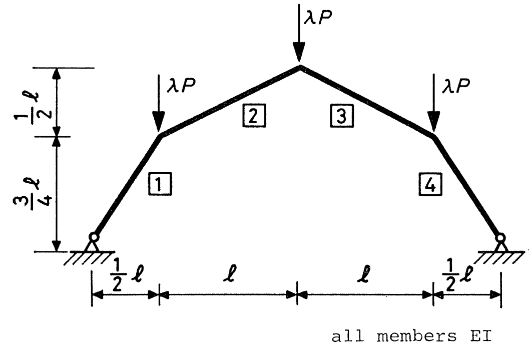

Fig. 10. Arch structure.

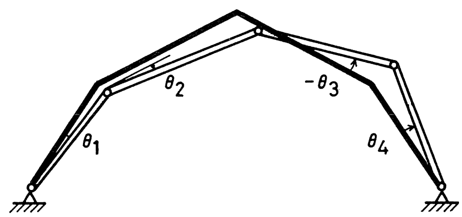

Fig. 11. Rotations of the rocker member model.

The structure is subdivided into four elements. The approximation model will then likewise comprise four rocker members; the rotations of these members will be designated by θ_1 to θ_4 (see fig. 11). The rotations θ_1 and θ_4 are chosen as degrees of freedom of the structure. The rotations of the rocker members 2 and 3 can be expressed in θ_1 and θ_4 by the following simple relations:

$$\theta_2 = -\theta_1 + \tfrac{1}{2}\theta_4$$

$$\theta_3 = \tfrac{1}{2}\theta_1 - \theta_4$$

The normal forces are:

$$N_1 = N_4 = -1.80\lambda P$$

$$N_2 = N_3 = -1.12\lambda P$$

Now consider an arbitrary member i which transmits a normal force N_i. If this member undergoes a rotation θ_i, a non-linear loading is produced, as indicated in fig. 12. By applying a similar consideration to all the members of the structure we can calculate the non-linear load for the structure as a whole, expressed in θ_1, θ_4 and λP (fig. 13).

By means of the linear elastic theory the displacements due to this load can be determined. We obtain:

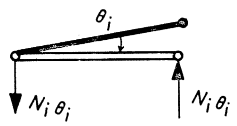

Fig. 12. Non-linear load on an arbitrary member.

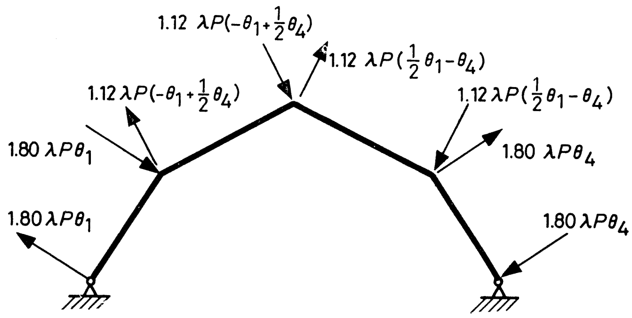

Fig. 13. Non-linear load due to θ_1 and θ_4.

$$\theta_1 = \{0.38(\lambda P\theta_1)+0.19(\lambda P\theta_4)\}l^2/EI$$

$$\theta_4 = \{0.19(\lambda P\theta_1)+0.38(\lambda P\theta_4)\}l^2/EI$$

whence the buckling values are obtained:

$$\lambda_1 = 1.75\frac{EI}{Pl^2} \quad \text{and} \quad \lambda_2 = 5.25\frac{EI}{Pl^2}$$

The buckling modes are respectively:

$$\frac{\theta_1}{\theta_4} = +1 \quad \text{and} \quad \frac{\theta_1}{\theta_4} = -1$$

The buckled shape associated with the lower buckling mode is shown in fig. 14.

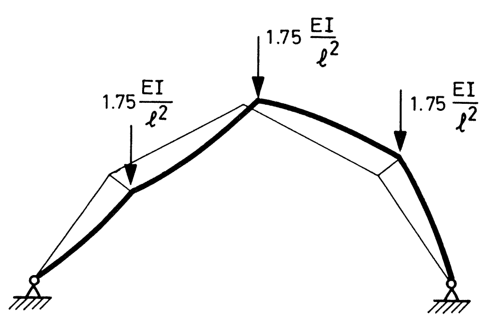

Fig. 14. Buckled shape associated with the smallest buckling load.

Lower bound analysis

For determining the lower bound the members 1 and 4 must be compared with 2 and 3.

Members 1 and 4:

$$N_i = -1.80P$$

$$P_{E_i} = \frac{\pi^2 EI}{(0.9l)^2}$$

$$\frac{-N_i}{P_{E_i}} = +0.147\frac{Pl^2}{EI}$$

Members 2 and 3:

$$N_i = -1.12P$$

$$P_{E_i} = \frac{\pi^2 EI}{(1.12l)^2}$$

$$\frac{-N_i}{P_{E_i}} = 0.142\frac{Pl^2}{EI}$$

The members 1 and 4 are the deciding ones; the lower bound thus becomes:

$$\frac{1}{\lambda_l} = \frac{1}{1.75}\left[\frac{Pl^2}{EI}\right] + 0.147\left[\frac{Pl^2}{EI}\right]$$

$$\lambda_l = 1.4\frac{EI}{Pl^2}$$

7. Concluding remarks

It is possible, with relatively little arithmetical effort, to obtain an upper and a lower bound for the elastic buckling load of two dimensional frames.

Adopting a reasonable subdivision of the structure into elements for the purpose of analysis, the order of magnitude of the difference between the upper and the lower bound is found to be between 5% and 25% of the buckling load.

For reasons already mentioned in the introduction there will be, in general, no need for higher accuracy.

If nevertheless for any particular structure a closer approximation is desired, clear indications for a suitable subdivision into elements can be obtained.

PART TWO: PROOF OF VALIDITY

Starting points

Consider a framed structure which is so loaded that only normal forces are developed in it. These normal forces are assumed to be compressive forces. This assumption will be used a number of times in giving the proof required. Strictly speaking, the results are therefore applicable only to those cases. It is furthermore assumed that the deformation due to normal force plays no part.

8. Rayleigh's principle

By means of Rayleigh's principle the buckling problem can be formulated as a minimalization problem (see, inter alia, ref. [3]). Determine the kinematicly permissible state of deformation of the frame for which the value of λ in the equation:

$$E = \lambda \cdot W \tag{20}$$

is a minimum. In this expression:

E = internal strain energy of the frame;
λ = a scale factor for the load;
W = external work done by the load in the case of a unit load ($\lambda = 1$).

The value found for λ is the buckling load factor λ_e.

The state of deformation for which $\lambda = \lambda_e$ is the associated buckled shape.

The internal strain energy E can be written as a summation comprising all the members. An arbitrary member in the deformed and in the non-deformed state is shown in fig. 15.

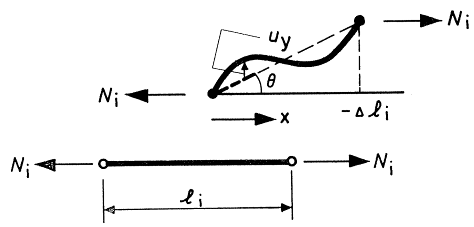

Fig. 15. Member i in the deformed and the non-deformed state.

The rigid translatory displacements of the member are not of importance; but the following are important:

θ = rigid rotation of the member;
u = function of x which described the deflection curve of the member in relation to the straight line connecting the displaced nodes (joints of the frame).

The strain energy can then be written as:

$$E = \Sigma \int_0^l \tfrac{1}{2} EI (u_{y,xx})^2 \, dx \tag{21}$$

The summation comprises all the members (subscripts relating to the members have been omitted for convenience). The formula is meaningful only within the context of a linear or second-order theory.

The external work W (work done by the outside forces) can also be written as a summation comprising all the members.

The load on the structure is of such a kind that nothing but normal forces are produced in it. Such a load can quite simply be replaced by a load scheme in which two equal but opposite axial forces act upon each member of the structure.

These forces are of such magnitude that the same pattern of normal forces is produced as is produced by the original load.

The external work can now simply be written as (see fig. 15):

$$W = \Sigma N_i \Delta l_i \tag{22}$$

where

$$\Delta l_i = \int_0^l -\tfrac{1}{2}(\theta + u_{y,x})^2 \, dx \tag{23}$$

N_i = normal force in member i

Formula (23) can be worked out:

$$-\Delta l_i = \int_0^l \tfrac{1}{2}\theta^2 \, dx + \int_0^l \tfrac{1}{2}u_{y,x}^2 \, dx + \int_0^l \theta u_{y,x} \, dx$$

Since θ is not a function of x:

$$-\Delta l_i = \tfrac{1}{2}\theta^2 l + \int_0^l \tfrac{1}{2}u^2{}_{y,x} dx + \theta \int_0^l u_{y,x} dx$$

The last integral:

$$\theta \int_0^l u_{y,x} dx = \theta \int_0^l du = \theta\{u(l) - u(0)\} = 0$$

The external work W for unit load $\lambda = 1$ can therefore finally be written as follows:

$$W = \Sigma - \tfrac{1}{2}N_i l\theta^2 + \Sigma[-\tfrac{1}{2}N_i \int u^2{}_{y,x} dx] \tag{24}$$

Summation comprises all the members; subscripts for l, θ and u have been omitted.

With the aid of (20), (21) and (24) the buckling problem can be formulated as follows:

For each member: so determine a compatible value of θ and a compatible deflection function u that λ in the equation:

$$\Sigma \int \tfrac{1}{2}EI(u_{y,xx})^2 \, dx = \lambda\{\Sigma - \tfrac{1}{2}N_i\theta^2 l + \Sigma \int -\tfrac{1}{2}N_i(u_{y,x})^2 \, dx\} \tag{25}$$

is a minimum.

On the assumption – as has been made here – that all N_i are negative, a positive value will always be obtained for λ. If there are also tensile forces acting in a structure, the problem becomes much more complicated. In such a case we may particularly wish to determine the smallest positive buckling load or to determine the smallest buckling load in the absolute sense. What is set forth below concerning upper and lower bounds will then no longer be strictly applicable. Yet even in those cases the upper bound approximation and the lower bound formula will usually be serviceable.

With the aid of formula (25) it is possible to find approximations for λ_e. On substituting an arbitrary state of displacement into (25) we find an upper bound for λ_e. In proportion as the state of displacement is in closer agreement with the actual buckled shape the value of λ will achieve a closer approximation to the buckling load.

9. Upper bound approximation

The approximate method presented in Part I can alternatively be written as a minimalization problem in accordance with Rayleigh's principle. For that purpose the simplest procedure is to base oneself on the physical model. The expression for the elastic energy of the model is the same as expression (21) for the original structure. The expression for the external work W is reduced, because of the rocker members, to:

$$W = \Sigma - \tfrac{1}{2} N_i l \theta^2$$

From this it follows at once that the approximation is an upper bound, because on the right-hand side of (25), since all $N_i < 0$, positive terms are cancelled and too large a value for λ will be found.

10. Maximum over-estimation by the upper bound approximation

Suppose that the exact buckled shape of the original structure is known. Substitution of this exact buckled shape into equation (25) will of course give the exact buckling load λ_e as the result:

$$\Sigma \int \tfrac{1}{2} EI u_{y,xx}^2 \mathrm{d}x = \lambda_e [\Sigma - \tfrac{1}{2} N_i \theta^2 l + \Sigma - \tfrac{1}{2} N_i \int u_{y,x}^2 \mathrm{d}x] \qquad (25)$$

with u and θ associated with the exact buckled shape.

Now substitute the same buckled shape into the Rayleigh equation for the model:

$$\Sigma \int \tfrac{1}{2} EI u_{y,xx}^2 = \lambda_u^* [\Sigma - \tfrac{1}{2} N_i \theta^2 l] \qquad (26)$$

The resulting value λ_u^* is naturally larger than the "exact" buckling load of the model λ_u:

$$\lambda_u^* > \lambda_u \quad \text{where} \quad \lambda_u = \text{upper bound approximation} \qquad (27)$$

From (25) and (26) we obtain:

$$\frac{1}{\lambda_e} - \frac{1}{\lambda_u^*} = \frac{\Sigma \int - \tfrac{1}{2} N_i u_{y,x}^2 \mathrm{d}x}{\Sigma \int \tfrac{1}{2} EI u_{y,xx}^2 \mathrm{d}x} \qquad (28)$$

On applying Rayleigh's principle to a single member pin-jointed at both ends, we find:

$$\int_0^l \tfrac{1}{2}EIu_{y,xx}^2 \,\mathrm{d}x \geqslant P_E \int_0^l \tfrac{1}{2}(u_{y,x})^2 \,\mathrm{d}x$$

where

$$P_E = \frac{\pi^2 EI}{l^2}$$

(the equality sign is valid only if u_y is a half sine wave).

Summation of this inequality comprising all the members:

$$\Sigma \int \tfrac{1}{2}EIu_{y,xx}^2 \,\mathrm{d}x \geqslant \Sigma P_{E_i} \int \tfrac{1}{2}(u_{y,x})^2 \,\mathrm{d}x$$

$$\geqslant \Sigma \left[\frac{P_{E_i}}{-N_i}\right] \int -\tfrac{1}{2}N_i u_{y,x}^2 \,\mathrm{d}x$$

$$\geqslant \left[\frac{P_{E_i}}{-N_i}\right]_{\min} \Sigma \int -\tfrac{1}{2}N_i u_{y,x}^2 \,\mathrm{d}x$$

(In the last two steps all N_i are assumed to be nonzero. This however is not essential).

Combining with (28):

$$\frac{1}{\lambda_e} - \frac{1}{\lambda_u^*} \leqslant \left[-\frac{N_i}{P_{E_i}}\right]_{\max}$$

and because of (27):

$$\frac{1}{\lambda_e} - \frac{1}{\lambda_u} \leqslant \left[\frac{-N_i}{P_{E_i}}\right]_{\max} \tag{29}$$

Formula (29) indicates the maximum over-estimation by the upper bound approximation. The lower bound formula (19) can then easily be proved with the aid thereof.

11. Proof of lower bound formula

The lower bound formula is:

$$\frac{1}{\lambda_l} = \frac{1}{\lambda_u} + \left[-\frac{N_i}{P_{E_i}}\right]_{\max}$$

With the aid of (29) we can eliminate λ_u from this formula, so that the following inequality is obtained for λ_l:

$$\frac{1}{\lambda_l} \geqslant \frac{1}{\lambda_e} - \left[\frac{-N_i}{P_{E_i}}\right]_{\max} + \left[\frac{-N_i}{P_{E_i}}\right]_{\max} = \frac{1}{\lambda_e}$$

or $\lambda_l \leqslant \lambda_e$, which constitutes the required proof.

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