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NON-LINEAR ANALYSIS OF CONCRETE MEMBERS

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NON-LINEAR ANALYSIS OF CONCRETE MEMBERS

Summary

This report deals with the calculation of the ultimate load of an axially compressed concrete member. It does not matter whether the member is straight or circular curved. Attention is mainly devoted to geometrical non-linearity (2nd order effect) and to physical non-linearity. The latter phenomenon is caused by the dependence of the bending stiffness on the state of load. The methods described offer a rather easy way of calculating the ultimate load. They are illustrated by two examples.

Acknowledgement

The analysis of concrete shell stability has become urgent following the recent growth in offshore activity. The amount of experimental data directly related to offshore construction was limited. Moreover there was a need for a theoretical analysis for the implosion pressure of concrete shells. The two methods of analysis described in this publication aim at satisfying that need.

Thanks are due to Ir. A. K. de Groot, Ir. W. J. Copier and Ir. F. B. J. Gijsbers, fellow-members of the Institute, for their constructive comments.

The author

Notation

(additional symbols of the appendices are not given here)

b	width of member
e	eccentricity of the normal load or deflection
e_0	initial deflection
e_t	second order deflection
f'_c	compressive strength of concrete
f_y	yield stress of reinforcement steel
l	length of member
m	number of half waves of the deflected shape
n	relative normal force $N/bt f'_c$
p	uniform pressure
t	total thickness of member
A, A'	cross-sectional area of reinforcement
E	modulus of elasticity
E_c	modulus of elasticity of concrete
E_{c0}	modulus of elasticity of concrete at the origin
G	geometrical factor
I	moment of inertia
M	bending moment
M_0	bending moment (1st order)
M_t	bending moment (2nd order)
N	normal force
N_{ac}	actual normal force
N_{cr}	critical normal force
N_{max}	maximum normal force
N_u	ultimate normal force
R	radius
S	bending stiffness
γ	safety margin
ε_a	steel strain
ε'_c	concrete compressive strain
ε'_u	ultimate compressive strain
θ	half the aperture of an arch
η	parameter, depending on θ
\varkappa	curvature
\varkappa_t	curvature (2nd order)
ν	Poisson's ratio
σ_a	steel stress
σ'_c	concrete compressive stress
σ'_{ct}	concrete compressive stress in extreme fibre
φ	parameter, depending on $l/R, t/R$ and m

Non-linear analysis of concrete members

1 Introduction

In calculating bending moments and deflections of an axially compressed concrete member we can, in general, distinguish between two cases. In the first case the bending moments are supposed not to be influenced by the deflections. This case is mostly referred to as “1st order”. In the second case, referred to as “2nd order”, the bending moments depend, apart from the axial load, on the magnitude of the deflections. This may be called “geometrical non-linear” too.

The considerations concerning 1st and 2nd order effects apply not only to simple columns, but also to circular curved beams or plates (rings and shells) with uniform radial load. In these cases there will occur a tangential force in the curved member which is quite analogous to the normal force in a straight member.

If the bending stiffness of the member is a constant, we may state that the 2nd order analysis is not a serious problem in most cases. But there could be difficulties, for example shells with varying edge conditions.

This report deals mainly with the complication that the bending stiffness of (reinforced) concrete is *not* a constant. The bending stiffness S is defined here as the bending moment M divided by the appropriate curvature \varkappa . A typical $M-\varkappa$ relation is given in Fig. 1.

In the following, a realistic approach to the 2nd order analysis is given. Two slightly different methods have been distinguished. The first is based on the assumption of an uncracked cross-section (Chapter 3). In Chapter 4 a more general method is developed. Both methods make use of the so-called critical load which is first dealt with in Chapter 2.

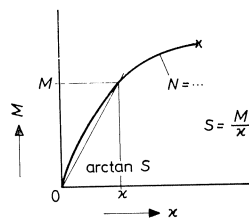


Fig. 1. Moment-curvature relation, typical.

2 Critical load

The critical load is defined as the load at which deflections, and therefore bending moments, increase indefinitely and when these deflections are initiated by an infinitely small lateral deflection of the member. For example the critical load N_{cr} of an axially compressed member with length l and constant stiffness $S(=EI)$ is given by:

$$N_{cr} = \frac{\pi^2 S}{l^2}$$

The critical load can be considered as a hypothetical one, for the initial deflection, which cannot be avoided in practice, causes an ultimate load N_u always less than N_{cr} . Nevertheless the magnitude of N_{cr} is of interest for it greatly influences the ultimate load N_u .

The critical load can be written as:

$$N_{cr} = S \cdot G \tag{1}$$

where G is a geometrical factor with the dimension of one per square unit length ($1/m^2$).

To obtain some idea of G , some specific cases are mentioned. See also Figs. 2 to 5.

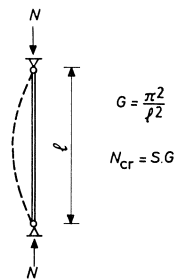


Fig. 2. Critical load of a hinged column.

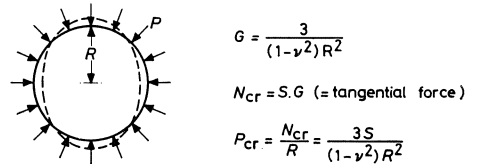


Fig. 3. Critical load of an infinitely long cylinder (cross-section shown).

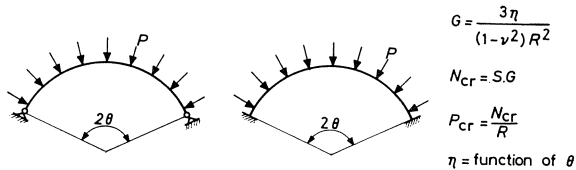


Fig. 4. Critical load of an infinitely long cylinder segment (cross-sections shown).

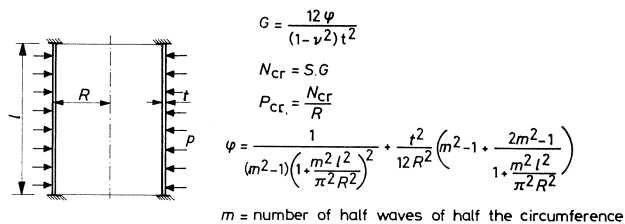


Fig. 5. Critical load of a cylinder with length l clamped at the edges.

- For a hinged column, length l , axially loaded:

$$G = \frac{\pi^2}{l^2}$$

- For an infinitely long cylinder with radius R under uniform radial pressure:

$$G = \frac{3}{(1-\nu^2)R^2}$$

- For an infinitely long cylinder segment with radius R and aperture 2θ , under radial pressure:

$$G = \frac{3\eta}{(1-\nu^2)R^2}$$

where η is a function of the aperture.

- For a cylinder with length l , radius R , thickness t , uniformly loaded:

$$G = \frac{12\varphi}{(1-\nu^2)t^2}$$

where φ is a function of l/R , t/R and the number m of waves of the deflected shape.

These expressions can be found in several well-known books on elastic stability, for example Timoshenko and Gere's "Theory of elastic stability".*

A very comprehensive collection of N_{cr} -values is given in "Handbook of structural stability".**

In the following treatment of the subject it is assumed that the geometrical factor G is known. For the cases mentioned above the parameters η and φ are given in Appendix A.

3 The uncracked cross-section

Assuming an uncracked section and neglecting possible reinforcement, the bending stiffness depends only on the (variable) modulus of elasticity E_c of the concrete and on the constant moment of inertia I .

According to the conventional approach, we have:

$$S = E_c \cdot I \tag{2}$$

* Int. Stud. Edition, McGraw-Hill, Tokyo.

** Editor: Column Research Committee of Japan; Corona Publishing Company Ltd., Tokyo.

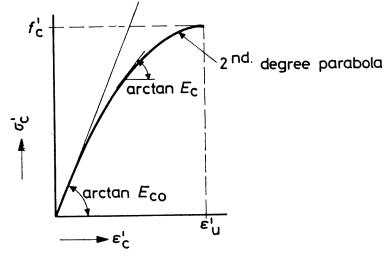


Fig. 6. Stress-strain diagram adopted for concrete.

For the stress-strain relation of concrete a 2nd degree parabola is adopted – Fig. 6. This relation can be written as:

$$\frac{\sigma'_c}{f'_c} = \frac{\varepsilon'_c}{\varepsilon'_u} \left(2 - \frac{\varepsilon'_c}{\varepsilon'_u} \right) \quad (3)$$

where

- σ'_c = compressive stress of concrete
- f'_c = compressive strength of concrete
- ε'_c = compressive strain of concrete
- ε'_u = ultimate strain of concrete

The modulus of elasticity is defined as the tangent modulus:

$$E_c = \frac{d\sigma'_c}{d\varepsilon'_c} = \frac{2f'_c}{\varepsilon'_u} \left(1 - \frac{\varepsilon'_c}{\varepsilon'_u} \right)$$

At the origin of the stress-strain relation:

$$E_{co} = \frac{2f'_c}{\varepsilon'_u}$$

We can now write:

$$E_c = E_{co} \left(1 - \frac{\varepsilon'_c}{\varepsilon'_u} \right) = E_{co} \sqrt{1 - \frac{\sigma'_c}{f'_c}} \quad (4)$$

It is emphasized that expression (4) is mainly introduced to account for a decreasing E_c -modulus with increasing stress level. For a given stress level however, the E_c -modulus is assumed to be a constant value across the section. In other words, there exists a linear stress distribution across the section.

Consider a rectangular section $b \cdot t$. The section is loaded by a normal force N and a 1st order moment $M_0 = N \cdot e_0$, where e_0 is the initial deflection. The 1st order compressive stress in the extreme fibre will be:

$$\sigma'_c = \frac{N}{bt} + \frac{M_0}{W} = \frac{N}{bt} + 6 \cdot \frac{Ne_0}{bt^2} = \frac{N}{bt} \left(1 + 6 \cdot \frac{e_0}{t} \right) \quad (5)$$

It is well-known that deflections and bending moments increase until certain limiting values are reached, belonging to a stable equilibrium (2nd order effect). These limiting values can be expressed as:

$$e_t = e_0 \frac{1}{1 - N/N_{cr}} \quad \text{and} \quad M_t = M_0 \frac{1}{1 - N/N_{cr}} \quad (6)$$

where t denotes the final value.

For a derivation of e_t and M_t , see Appendix B.

Note that these expressions are based on the assumption that all sections of the considered member have the same bending stiffness.

The extreme fibre stress will be:

$$\sigma'_{ct} = \frac{N}{bt} + 6 \cdot \frac{Ne_t}{bt^2} = \frac{N}{bt} \left(1 + 6 \cdot \frac{e_0}{t} \cdot \frac{1}{1 - N/N_{cr}} \right) \quad (7)$$

Using:

$$E_c = E_0 \sqrt{1 - \frac{\sigma'_c}{f'_c}} \quad \text{and} \quad N_{cr} = (E_c I) \cdot G$$

the bending stiffness is realistically taken into account.

But what concrete stress σ'_c should be used in the expression of E_c ?

A safe approximation is to use the maximum compressive stress that occurs anywhere in the member. Then we get the lowest, thus safe values of E_c and N_{cr} .

It is more likely that the mean stress in the section should be taken. We may assume that the overall behaviour depends more on some mean bending stiffness rather than on a local low value.

In Appendix C it is deduced that a realistic value of E_c can be based on the stress at the centroidal axis corresponding to a parabolic stress distribution across the section, see Fig. 7.

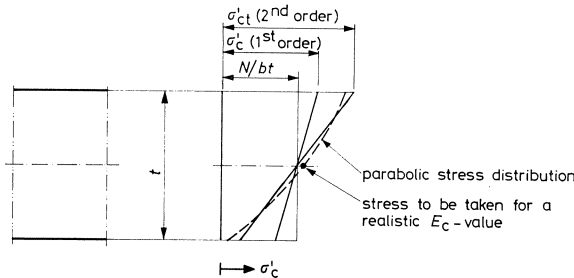


Fig. 7. Linear and parabolic stress distributions.

The appropriate expression (Eq. (13) of Appendix C) depends on $N/bt f'_c$ and the relative eccentricity e/t of the normal force with respect to the centroidal axis. That expression is not a very simple one. But for the purpose of calculating the ultimate load, in the appendix a very good approximation is derived, stating:

$$E_c = E_{co} \sqrt[3]{\frac{3}{4} \left(1 - \frac{N}{bt \cdot f'_c} \right)} \quad (8)$$

which is independent of the eccentricity!

On the assumption that the ultimate load N_u is reached when σ'_{ct} equals the concrete compressive strength f'_c , N_u can be taken from:

$$f'_c = \frac{N_u}{b \cdot t} \left(1 + 6 \cdot \frac{e_0}{t} \cdot \frac{1}{1 - N_u/N_{cr}} \right) \quad (9)$$

where

$$N_{cr} = E_{co} I G \sqrt[3]{\frac{3}{4} \left(1 - \frac{N_u}{bt \cdot f'_c} \right)}$$

This requires the solution of a 3rd degree equation in N_u . The solution for N_u can very conveniently be done graphically by calculating values of σ'_{ct} as a function of some values of N .

The method is illustrated by the following example. Given a rectangular cross-section with:

$$\begin{aligned} b &= 1.00 \text{ m} \\ t &= 0.55 \text{ m} \\ E_{co} &= 29000 \text{ N/mm}^2 \\ f'_c &= 40 \text{ N/mm}^2 \\ G &= 68.4 \cdot 10^{-3} \text{ 1/m}^2 \\ e_0 &= 0.03 \text{ m} \\ N_{ac} &= 6 \text{ MN (actual load)} \end{aligned}$$

What is the safety margin γ against failure?

The 1st order stress, according to Eq. (5) is:

$$\sigma'_c = \frac{N}{1.00 \cdot 0.55} \left(1 + 6 \cdot \frac{0.03}{0.55} \right) = 2.41 \cdot N \quad \text{MN/m}^2 \quad (\equiv \text{N/mm}^2) \quad (10a)$$

Modulus of elasticity, according to Eq. (8):

$$E_c = 29000 \sqrt[3]{\frac{3}{4} \left(1 - \frac{N}{22} \right)} \quad \text{MN/m}^2$$

Critical load:

$$N_{cr} = (E_c I) \cdot G = E_c \cdot \frac{1}{12} b t^3 \cdot G$$

$$N_{cr} = 27.54 \sqrt{\frac{3}{4} \left(1 - \frac{N}{22}\right)} \quad \text{MN} \quad (10b)$$

Second order stress, according to Eq. (7):

$$\sigma'_{ct} = \frac{N}{1.00 \cdot 0.55} \left(1 + 6 \cdot \frac{0.03}{0.55} \cdot \frac{1}{1 - N/N_{cr}}\right) \quad (10c)$$

Some calculated values are listed below.

N (MN)	σ'_c (N/mm ²)	N_{cr} (MN)	σ'_{ct} (N/mm ²)
6	14.5	20.3	16.0
8	19.3	19.0	22.8
10	24.1	17.6	31.9
11	26.6	16.9	38.8
12	29.0	16.1	50.0

Plotting σ'_{ct} as a function of N , we obtain Fig. 8. The vertical line, indicated by N_{cr}^* , is the asymptote of the calculated relation. Its value follows from the condition $N = N_{cr}^*$. For this specific case, from Eq. (10b):

$$N_{cr}^* = 27.54 \sqrt{0.75 - 0.75 N_{cr}^*/22} \quad \text{or} \quad N_{cr}^* = 14.2 \text{ MN}$$

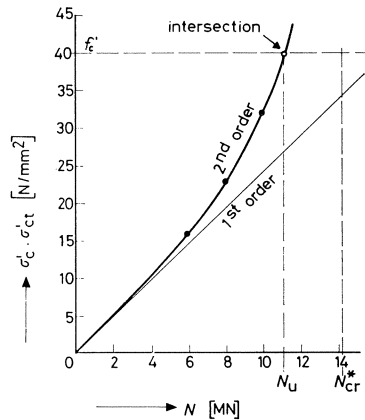


Fig. 8. Determination of N_u .

The intersection of the $\sigma'_{cr}-N$ relation with the concrete strength $f'_c = 40 \text{ N/mm}^2$ gives $N_u = 11.1 \text{ MN}$.

The safety against failure, defined as the ratio N_u/N_{ac} , is:

$$\gamma = \frac{11.1}{6} = 1.85$$

This method gives a clear view of what is happening when the load increases. It is evident that failure is always caused by reaching the concrete strength before the critical load causes instability, unless the initial deflection is zero. So it is not correct to define the safety as the ratio N_{cr}^*/N_{ac} .

Because of $\sigma'_{cr}(\text{minimum}) = 2 \cdot \sigma'_{cr}(\text{mean}) - f'_c = 0.4 \text{ N/mm}^2 (> 0)$, the condition of an uncracked section is satisfied. But we can be sure that in the case of a greater initial deflection the assumption of a non-cracked section is not correct.

4 General method

The determination of the ultimate load in the case of an arbitrary cross-section (cracked, reinforced) needs a more general method. Such a method is described here. The $M-N-\kappa$ relations, reflecting all material properties, are assumed to be known. Furthermore, use is made of the M_u-N_u interaction diagram which can be deduced from the $M-N-\kappa$ relations. Typical diagrams are given in the Figs. 9a and 9b.

An $M-N-\kappa$ diagram gives, at a certain normal load N , the *internal* relation between the bending moment and the curvature belonging to the considered section. The bending stiffness is taken as $S = M/\kappa$.

It is also possible, as a consequence of the 2nd order effect, to speak of an *external* relation between the 2nd order moment M_t and the 2nd order curvature κ_t . This depends on the geometry of the member.

The external relation is derived from the expression (6):

$$M_t = M_0 \cdot \frac{1}{1 - N/N_{cr}}$$

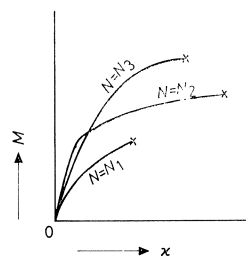


Fig. 9a. Typical $M-N-\kappa$ relations.

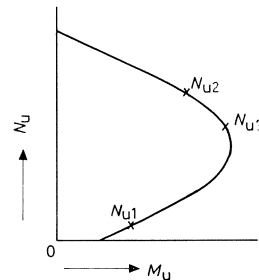


Fig. 9b. M_u-N_u interaction diagram.

With $N = M_0/e_0$ and $N_{cr} = S \cdot G$ follows:

$$M_t = M_0 \cdot \frac{1}{1 - M_0/e_0 G S}$$

or

$$M_t = M_0(1 + M_t/e_0 G S)$$

By substituting $S = M_t/\kappa_t$ we get:

$$M_t = M_0(1 + \kappa_t/e_0 G) \tag{11}$$

Equation (11) gives the external relation between M_t and κ_t .

Both relations must be satisfied to obtain equilibrium. So the desired value of M_t is found by the intersection of the relation $M = M_0(1 + \kappa/e_0 G)$ and the given $M-\kappa$ relation.

A typical graph is presented in Fig. 10.

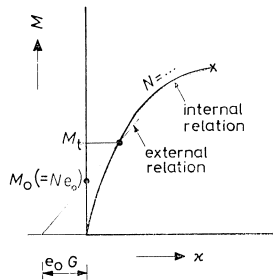


Fig. 10. Graphical determination of M_t .

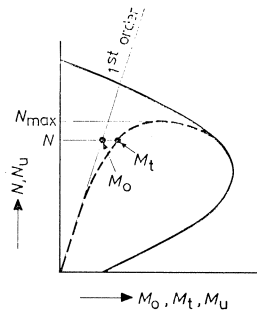


Fig. 11. Graphical determination of N_{max} .

Repeating this procedure for several values of the 1st order moment $M_0 (= N \cdot e_0)$, we find a family of $N - M_t$ relations. Plotting these relations in the given $M_u - N_u$ interaction diagram we readily get the extreme value N_{max} , being the load at which failure is initiated. See Fig. 11.

It is of interest to note that N_{max} , at the top of the $N - M_t$ curve, corresponds with the situation where the line of external relation is the tangent line to the $M - \kappa$ curve. This situation reflects a state of unstable equilibrium. Only values smaller than N_{max} cause stable equilibrium. Beyond N_{max} equilibrium is not possible. Assuming that N is gradually increased, a slightly greater value of N_{max} will cause a considerable increase of the deflection and bending moment until failure occurs at N_u equal to N_{max} .

The method is illustrated by means of the following example.

The same dimensions and characteristics of the previous example of Chapter 3 are given. In addition symmetrical reinforcement is considered. Data:

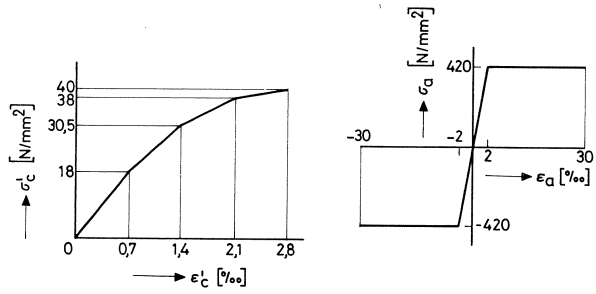


Fig. 12. Stress-strain relations adopted for concrete and steel.

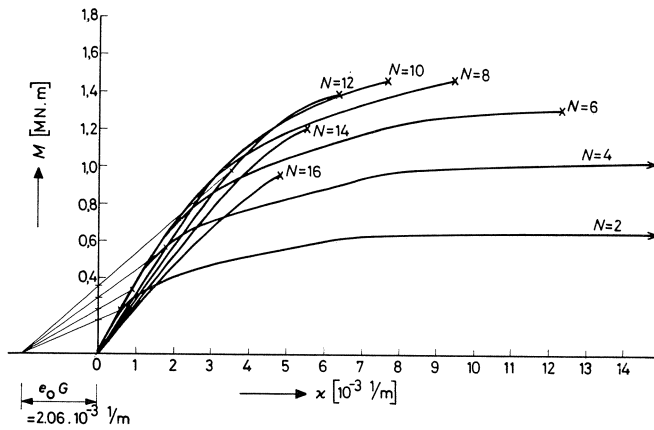


Fig. 13. Calculated $M-N-z$ diagram.

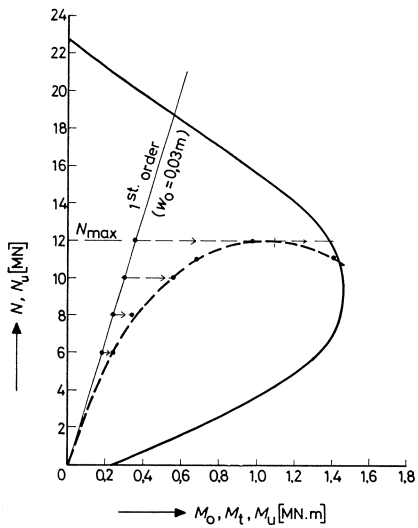


Fig. 14. Calculated $M_u - N_u$ interaction diagram.

$$\begin{aligned}
b &= 1.00 \text{ m} \\
t &= 0.55 \text{ m} \\
G &= 68.4 \cdot 10^{-3} \text{ l/m}^2 \\
e_0 &= 0.03 \text{ m} \\
N_{ac} &= 6 \text{ MN} \\
E_{co} &= 29000 \text{ N/mm}^2 \\
f'_c &= 40 \text{ N/mm}^2 \\
A &= A' = 0.002bt \\
f_y &= 420 \text{ N/mm}^2
\end{aligned}$$

The M - N - κ diagrams (Fig. 13) and the M_u - N_u interaction diagram (Fig. 14) are calculated by means of a computer. The stress-strain relations of concrete and steel which are used, are drawn in Fig. 12.

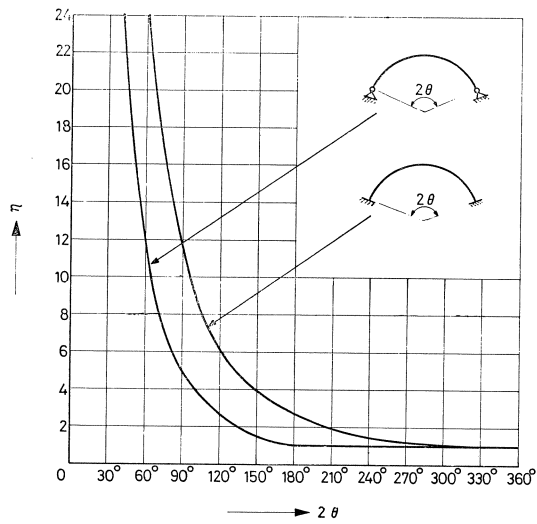
From Fig. 14 follows the failure load $N_{\max} = 12 \text{ MN}$ with the corresponding 2nd order moment $M_t = 1.1 \text{ MN.m}$. The safety factor is:

$$\gamma = \frac{N_{\max}}{N_{ac}} = \frac{N_u}{N_{ac}} = \frac{12}{6} = 2$$

It is not surprising that this safety is 8% greater than that found in Chapter 3: the reinforcement will cause a decrease in the concrete stresses and an increase in the bending stiffness.

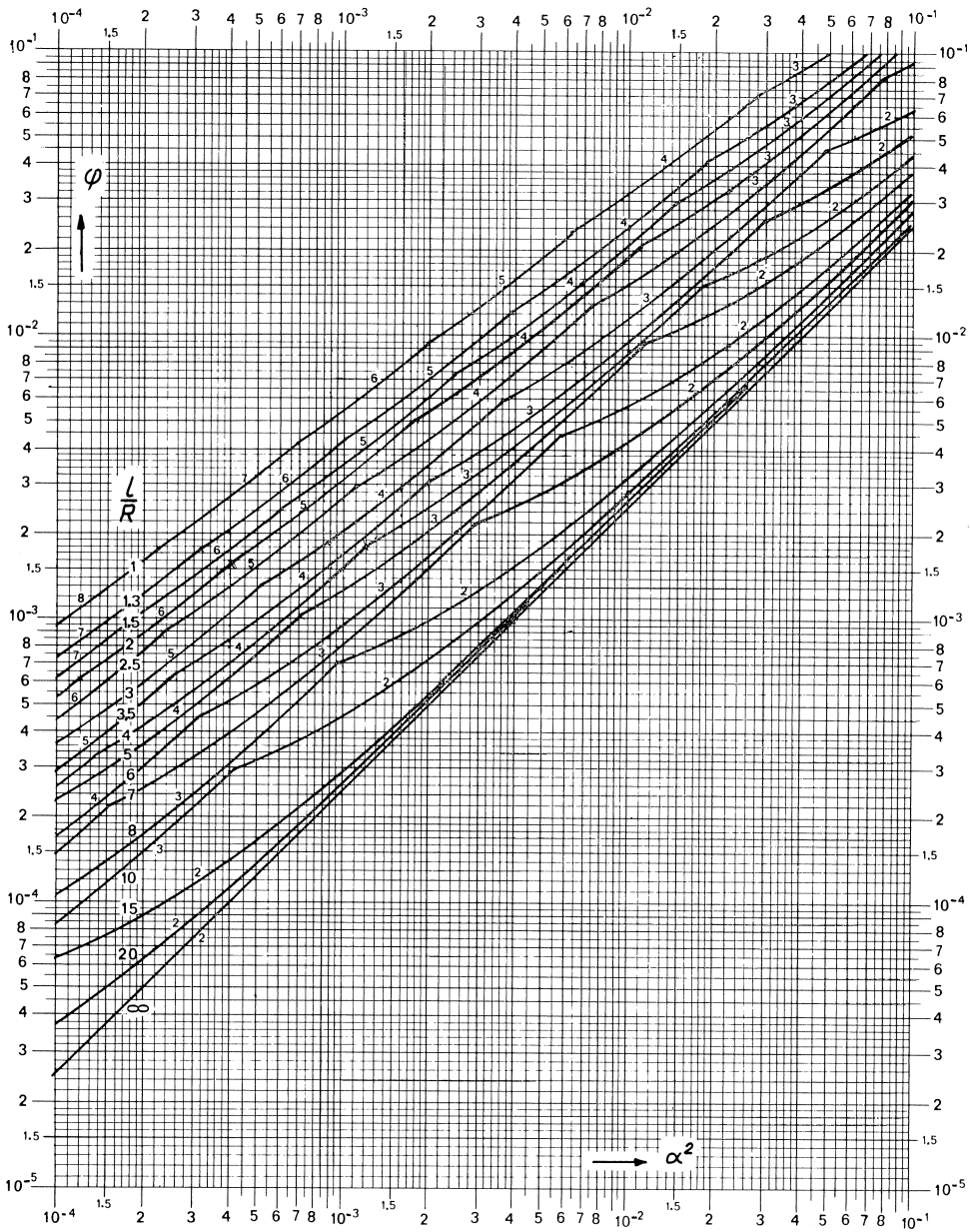
APPENDIX A.1

The parameter η as a function of the aperture 2θ



APPENDIX A.2

Graph of the parameter φ



$$\varphi = \frac{1 - \nu^2}{(m^2 - 1) \left(1 + \frac{m^2 l^2}{\pi^2 R^2} \right)^2} + \frac{1}{2} \alpha^2 \left(m^2 - 1 + \frac{2m^2 - 1 - \nu}{1 + \frac{m^2 l^2}{\pi^2 R^2}} \right)$$

$$\alpha = l/R$$

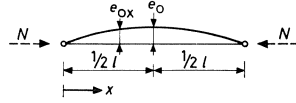
m = number of buckling waves (small digits)

$$\nu = 0.2$$

APPENDIX B

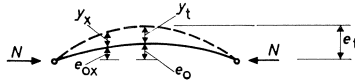
Determination of e_t and M_t

Given a member with length l and deflection $e_{0x} = e_0 \sin \frac{\pi x}{l}$.



The load N will cause an additional deflection y_x . The total deflection becomes $e_x = e_{0x} + y_x$, which is assumed to be a sine curve. So $e_x = e_t \cdot \sin \frac{\pi x}{l}$ and

$$y_x = e_x - e_{0x} = (e_t - e_0) \sin \frac{\pi x}{l} \quad (1)$$



In general:

$$M_x = -EI \cdot \frac{d^2 y_x}{dx^2}$$

From Eq. (1):

$$\frac{d^2 y_x}{dx^2} = -(e_t - e_0) \frac{\pi^2}{l^2} \sin \frac{\pi x}{l}$$

So:

$$M_x = EI \left((e_t - e_0) \cdot \frac{\pi^2}{l^2} \cdot \sin \frac{\pi x}{l} \right) \quad (2)$$

From the external load:

$$M_x = N \cdot e_x = N \cdot e_t \cdot \sin \frac{\pi x}{l} \quad (3)$$

From Eqs. (2) and (3):

$$(e_t - e_0) \cdot \frac{\pi^2 EI}{l^2} = N \cdot e_t$$

With:

$$N_{cr} = \frac{\pi^2 EI}{l^2}$$

follows:

$$e_t = e_0 \frac{N_{cr}}{N_{cr} - N} = e_0 \cdot \frac{1}{1 - N/N_{cr}}$$

Maximum moment:

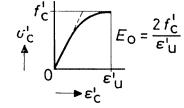
$$M_t = N \cdot e_t = N \cdot e_0 \cdot \frac{1}{1 - N/N_{cr}} = M_0 \frac{1}{1 - N/N_{cr}}$$

APPENDIX C

Determination of a realistic E_c -value

Assumptions

- Parabolic σ'_c - ε'_c diagram of concrete:
- Linear strain distribution across the section.
- All strains remain positive (compression).
- Reinforcement is neglected.



See also Fig. C-1.

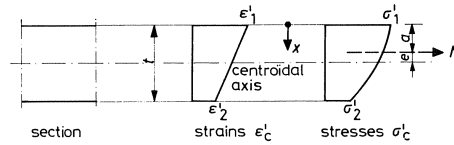


Fig. C-1.

Determination of N and e as functions of the strains

Strains:

$$\varepsilon'_x = \varepsilon'_1 - \frac{x}{h_t}(\varepsilon'_1 - \varepsilon'_2) \quad (1)$$

Stresses:

$$\sigma'_x = f'_c \cdot \frac{\varepsilon'_x}{\varepsilon'_u} \left(2 - \frac{\varepsilon'_x}{\varepsilon'_u} \right) \quad (2)$$

Normal force:

$$N = \int_0^t \sigma'_x b dx = \frac{1}{3} \frac{bt \cdot f'_c}{\varepsilon'^2_u} \{ 3\varepsilon'_u(\varepsilon'_1 + \varepsilon'_2) - (\varepsilon'_1 + \varepsilon'_2)^2 + \varepsilon'_1 \varepsilon'_2 \} \quad (3)$$

Distance a :

$$a = \frac{\int_0^t \sigma'_x b x dx}{N}$$

With

$$\int_0^t \sigma'_x b x dx = \frac{1}{12} \frac{bt^2 f'_c}{\varepsilon'^2_u} \{ 4\varepsilon'_u(\varepsilon'_1 + 2\varepsilon'_2) - (\varepsilon'_1 + \varepsilon'_2)^2 - 2\varepsilon'^2_2 \}$$

follows:

$$a = \frac{1}{4t} \cdot \frac{\{ 4\varepsilon'_u(\varepsilon'_1 + 2\varepsilon'_2) - (\varepsilon'_1 + \varepsilon'_2)^2 - 2\varepsilon'^2_2 \}}{\{ 3\varepsilon'_u(\varepsilon'_1 + \varepsilon'_2) - (\varepsilon'_1 + \varepsilon'_2)^2 + \varepsilon'_1 \varepsilon'_2 \}} \quad (4)$$

Eccentricity: $e = \frac{1}{2}t - a$ or:

$$e = \frac{1}{4}t \cdot \frac{\{2\varepsilon'_u(\varepsilon'_1 - \varepsilon'_2) - \varepsilon'^2_1 + \varepsilon'^2_2\}}{\{3\varepsilon'_u(\varepsilon'_1 + \varepsilon'_2) - (\varepsilon'_1 + \varepsilon'_2)^2 + \varepsilon'_1\varepsilon'_2\}} \quad (5)$$

With the Eqs. (3) and (5) the load N and the eccentricity e are known as functions of the strains. Because of $M = N \cdot e$ the bending moment M will be:

$$M = \frac{1}{12} \cdot \frac{bt^2 f'_c}{\varepsilon'^2_u} (\varepsilon'_1 - \varepsilon'_2) (2\varepsilon'_u - \varepsilon'_1 - \varepsilon'_2) \quad (6)$$

Capacity of the cross-section

This capacity will be reached when $\varepsilon'_1 = \varepsilon'_u$. Substituting $\varepsilon'_1 = \varepsilon'_u$ in Eq. (5) we find:

$$e = \frac{1}{4}t \cdot \frac{\varepsilon'^2_u - 2\varepsilon'_u\varepsilon'_2 + \varepsilon'^2_2}{2\varepsilon'^2_u + 2\varepsilon'_u\varepsilon'_2 - \varepsilon'^2_2}$$

Solution of ε'_2 :

$$\varepsilon'_2 = \varepsilon'_u \left\{ 1 - \sqrt{\frac{12e}{4e+t}} \right\} \quad (7)$$

Substitution of Eq. (7) into Eq. (3) gives:

$$N (= N_u) = bt f'_c \frac{1}{1 + \frac{t}{4e}} \quad (8)$$

Bending stiffness and modulus of elasticity

The curvature is:

$$\kappa = \frac{\varepsilon'_1 - \varepsilon'_2}{t} = \frac{M}{S}$$

So:

$$S = \frac{M \cdot t}{\varepsilon'_1 - \varepsilon'_2}$$

From Eq. (6) follows:

$$S = \frac{1}{12} bt^3 \cdot \frac{2f'_c}{\varepsilon'_u} \left(1 - \frac{\varepsilon'_1 + \varepsilon'_2}{2\varepsilon'_u} \right) \quad (9)$$

Because of the assumption of an uncracked section, we may state $S = E_c I$, where $I = \frac{1}{12} b t^3$.

With

$$\frac{2f'_c}{\varepsilon'_u} = E_{co}$$

Eq. (9) becomes:

$$E_c = E_{co} \left(1 - \frac{\bar{\varepsilon}}{\varepsilon'_u} \right) \quad (10)$$

where $\bar{\varepsilon} = \frac{1}{2}(\varepsilon'_1 + \varepsilon'_2) =$ mean strain (at centroidal axis).

From Eq. (2) can be derived:

$$1 - \frac{\bar{\varepsilon}}{\varepsilon'_u} = \sqrt{1 - \frac{\sigma'_{ca}}{f'_c}}$$

So:

$$E_c = E_{co} \sqrt{1 - \frac{\sigma'_{ca}}{f'_c}} \quad (10)$$

where $\sigma'_{ca} =$ concrete compressive stress at the centroidal axis (parabolic stress distribution).

Using the notation:

$$\frac{E_c}{E_{co}} = 1 - \alpha$$

where

$$\alpha = \frac{\bar{\varepsilon}}{\varepsilon'_u}$$

we have to determine α as a function of N and e .

Putting $\varepsilon'_1 + \varepsilon'_2 = 2\bar{\varepsilon}$ and $\varepsilon'_1 - \varepsilon'_2 = \varkappa t$, it follows $\varepsilon'_1 \varepsilon'_2 = \bar{\varepsilon}^2 - (\frac{1}{2}\varkappa t)^2$. Substitution into Eqs. (3) and (5) gives:

$$N = \frac{1}{3} \frac{b t f'_c}{\varepsilon_u'^2} \{ 6\varepsilon'_u \bar{\varepsilon} - 3\bar{\varepsilon}^2 - (\frac{1}{2}\varkappa t)^2 \}$$

$$e = \frac{1}{4} t \cdot \frac{2(\varepsilon'_u - \bar{\varepsilon})\varkappa t}{6\varepsilon'_u \bar{\varepsilon} - 3\bar{\varepsilon}^2 - (\frac{1}{2}\varkappa t)^2}$$

Using

$$\alpha = \frac{\bar{\varepsilon}}{\varepsilon'_u}, \quad \beta = \frac{\varkappa t}{2\varepsilon'_u} \quad \text{and} \quad n = \frac{N}{b t f'_c}$$

we get:

$$3n = 6\alpha - 3\alpha^2 - \beta^2$$

$$3n \frac{e}{t} = \beta(1 - \alpha)$$

Solution of these equations gives:

$$\alpha = 1 - \sqrt{\frac{1}{2}(1-n) \left[1 + \sqrt{1 - 12 \left(\frac{n}{1-n} \right)^2 \left(\frac{e}{t} \right)^2} \right]} \quad (11)$$

$$\beta = \frac{3n \frac{e}{t}}{1 - \alpha} \quad (12)$$

The realistic modulus of elasticity is thus given by:

$$E_c = E_{co} \sqrt{\frac{1}{2}(1-n) \left[1 + \sqrt{1 - 12 \left(\frac{n}{1-n} \right)^2 \left(\frac{e}{t} \right)^2} \right]} \quad (13)$$

Furthermore, there are two conditions to be satisfied: $\varepsilon'_1 \leq \varepsilon'_u$ and $0 \leq \varepsilon'_2 \leq \varepsilon'_1$.

For the boundary conditions can easily be derived:

$$\alpha = 1 - \beta = 1 - \frac{1}{2} \sqrt{\frac{12e}{4e+t}} \quad \text{if } \varepsilon'_1 = \varepsilon'_u \quad (14)$$

and

$$\alpha = \beta = \frac{t-6e}{t-4e} \quad \text{if } \varepsilon'_2 = 0 \quad (15)$$

So, with $\frac{E_c}{E_{co}} = 1 - \alpha$:

$$\frac{E_c}{E_{co}} = \frac{1}{2} \sqrt{\frac{12e}{4e+t}} \quad \text{if } \varepsilon'_1 = \varepsilon'_u \quad (16)$$

$$\frac{E_c}{E_{co}} = \frac{2e}{t-4e} \quad \text{if } \varepsilon'_2 = 0 \quad (17)$$

The Eqs. (13), (16) and (17) are graphically presented in Fig. C-2.

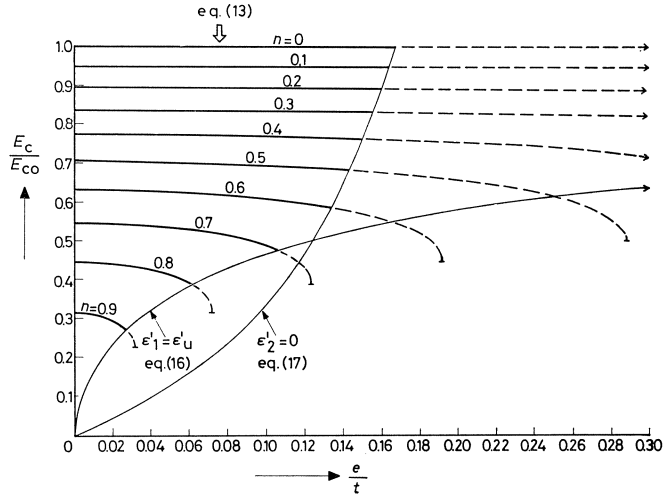


Fig. C-2. E_c/E_{co} as a function of $n(=N/bt'f'_c)$ and e/t .

Modulus of elasticity at ultimate ($\epsilon'_1 = \epsilon'_u$)

From the condition $\epsilon'_1 = \epsilon'_u$ it has been found that $1 - \alpha = \beta$. So, with Eq. (12) follows:

$$1 - \alpha = \sqrt{3n \frac{e}{t}} \quad (18)$$

The modulus of elasticity at ultimate is now found by equating Eq. (13) to Eq. (18). Thus:

$$\sqrt{\frac{1}{2}(1-n) \left[1 + \sqrt{1 - 12 \left(\frac{n}{1-n} \right)^2 \left(\frac{e}{t} \right)^2} \right]} = \sqrt{3n \frac{e}{t}}$$

From this expression:

$$\frac{e}{t} = \frac{1-n}{4n}$$

hence:

$$E_c = E_{co} \cdot \frac{1}{2} \sqrt{3} \cdot \sqrt{1-n} = E_{co} \sqrt{\frac{3}{4} \left(1 - \frac{N}{bt'f'_c} \right)} \quad (19)$$

In Eq. (19) the modulus of elasticity is expressed as a function of the relative normal force $n(=N/bt'f'_c)$ and independent of the eccentricity. This expression is easier to use than Eq. (13) and gives the correct E_c -value when $\epsilon'_1 = \epsilon'_u$. Eq. (19) is recommended when calculating the capacity (or N_u) of the cross-section.