## Preface

Inspired by the building of concrete offshore structures in the North Sea the Industrial Council for Oceanology (IRO) commisioned in 1975 the Institute TNO for Building Materials and Building Structures (IBBC-TNO) to undertake research on the subject fatigue of concrete. Project Group III-3 of the Stupoc (Steering Group on Offshore Structure Problems) has been entrusted with the routine supervision on that research project.

The Project Group was constituted as follows:

Ir. J. C. Slagter, Chairman
Ir. C. v.d. Fliert/Drs. J. Nieuwenhuis
Ing. A. C. van Riel
Ir. Ch. J. Vos

The experimental research comprised the testing of ca. 350 cylinders of plain concrete both with constant-amplitude loadings as with program loadings. The main object of the experimental research was to verify the validity of Miner's damage rule for plain concrete in compression.

In 1977 the Netherlands Committee for Concrete Research (CUR) decided to finance a further research project on the subject fatigue of concrete.

The IRO-fatigue project served as a starting point for this research. The CUR-committee C33 "Fluctuating loads" was charged with the routine supervision.

The Committee was constituted as follows:

Ir. W. Colenbrander, Chairman
Ir. P. Eggermont, Secretary
Ir. A. J. M. Siemes, Technical secretary
Ir. J. P. Coppin
Ir. F. F. M. de Graaf
Ir. H. A. Körmeling
Prof. Ir. H. Lambotte
Ir. J. van Leeuwen
Ir. J. H. A. M. Vrencken
Ir. P. H. Zaalberg
Ir. D. Zijp
Ir. J. C. Slagter, Mentor

The experimental work carried out at the IBBC-TNO comprised the testing of ca. 450 cylinders of plain concrete both with constant-amplitude loadings as with variable-amplitude loadings with the object of a further verification of Miner's rule. The main part of the results is included in this report.

The research at IBBC-TNO was carried out by Ir. J. van Leeuwen and Ir. A. J. M. Siemes who are also the authors of this report.

# MINER'S RULE WITH RESPECT TO PLAIN CONCRETE


## Summary

As usual with other materials the constant-amplitude test (Wöhler) is adopted as the criterion for the fatigue sensitivity of concrete. In this test the number of load repetitions $N_i$ the material can stand before failure occurs, is determined.
In general stresses vary in a more erratical way. As from tests no relation is knwon between this kind of loadings and the service life of a structure, Miner's rule is adopted for predicting this life on basis of constant-amplitude tests. According to this rule failure will occur if the following condition is satisfied:

$$M = \sum_{i=1}^{c} \frac{1}{N_i} = 1$$

where $c$ is the number of stress repetitions during the service life.
For plain concrete loaded in compression, IBBC-TNO have started a verification of this damage rule, by executing ca 385 constant-amplitude tests at several stress levels, ca 100 program loading tests and ca 180 variable-amplitude tests with several probability density functions for extreme values of the stresses.

The number of stress repetitions $N_i$ in a constant-amplitude test proved to be a stochastic value with a logarithmic-normal distribution. Because of this aspect the Miner number will not have the deterministic value one, but it will also be stochastic with a logarithmic-normal distribution. A formula is given for the median value and the standard deviation of $M$ that can be expected on basis of the dispersion in a constant-amplitude test.

From the program loading tests and the variable amplitude tests the Miner number proved to have a logarithmic-normal distribution with in general a median value less than one.

So it can be concluded that program loadings and variable-amplitude loadings are more damaging than can be expected from constant-amplitude tests.
As the Miner number is not deterministic a semi-probabilistic design procedure will be given. This procedure is based on the experience, gained in the investigation.

# NOMENCLATURE

| | |
|---|---|
| $c$ | number of stress cycles (loading) |
| $F_d$ | design loading |
| $F_k$ | characteristic loading |
| $f'_{bu}$ | static cylinder compressive strength |
| $f_d$ | design strength |
| $f_k$ | characteristic strength |
| $i$ | index indicating a certain stress level |
| $M$ | Miner sum or Miner number |
| $M_{Fd}$ | Miner sum caused by the design loading |
| $M_{Fk}$ | Miner sum caused by the characteristic loading |
| $M_{fd}$ | Miner number expressed as a design strength |
| $M_{fk}$ | Miner number expressed as a characteristic strength |
| $m(\log M)$ | mean value of $\log M$ |
| $m(M)$ | median value of $M$; |
| | according to a logarithmic-normal distribution $m(M) = 10^{m(\log M)}$ |
| $m(\log N_i)$ | mean value of $\log N_i$ |
| $m(N_i)$ | median value of $N_i$; |
| | according to a logarithmic normal distribution $m(N_i) = 10^{m(\log N_i)}$ |
| $m(\sigma)$ | mean stress in a variable amplitude test |
| $m(\hat{\sigma})$ | mean value of the stress amplitudes |
| $N_i$ | number of stress cycles giving failure in a constant-amplitude test at stress level $i$ |
| $n$ | number of stress cycles |
| $n_i$ | number of stress cycles in a block of a program loading |
| $R$ | stress ratio: $R = \sigma'(\min_i)/\sigma'(\max_i)$ |
| $R_d$ | design resistance |
| $S_d$ | design effect of the loading |
| $s(f'_{bu})$ | standard deviation of $f'_{bu}$ |
| $s(\log M)$ | standard deviation of $\log M$ |
| $s(M)$ | standard deviation of $M$, according to a logarithmic-normal distribution |
| $s(\log N_i)$ | standard deviation of $\log N_i$ |
| $s(\hat{\sigma})$ | standard deviation of the amplitudes in a variable-amplitude test |
| $w$ | dispersion range |
| | |
| $\beta$ and $\beta^*$ | reliability index |
| $\gamma_f$ | load factor |
| $\gamma_m$ | material factor |
| $\sigma'(\max_i)$ | maximum compressive stress |
| $\sigma'(\min_i)$ | minimum compressive stress |
| $\hat{\sigma}$ | stress amplitude |

**Notice**

**In this paper a distinction has been made between mean value and median value:**

**– mean value is the arithmetic mean;**

**– median value corresponds with a probability of 50%.**

**For symmetric probability density functions, like the normal distribution, the mean and the median have the same value.**

**For asymmetric probability density functions, like the logarithmic-normal distribution, the mean and the median have different values.**

# Miner's rule with respect to plain concrete

## 1 Introduction

The load variations to which concrete structures are subjected often display an erratic behaviour in relation to time. Besides, this behaviour is unpredictable, so that these loads should be treated as stochastic quantities.

Loads due to wind, wave motion, sea currents, earthquakes and traffics are of this kind. As a result of such loads, but also in consequence of the dynamic behaviour of a structure, stress variations may occur in the structural material which are likewise stochastic and of an erratic character in relation to time. These stress variations may in course of time cause fatigue failure. With the aid of constant-amplitude test (Wöhler test) an indication is obtained as to the extent to which a material is susceptible to fatigue. In this test (see Fig. 1) the stress alternates with a constant amplitude and constant frequency about a constant mean stress; the test is in itself therefore not sufficient to serve as a basis for predicting the lifetime of a construction subject to loads which behave erratically in relation to time (see example in Fig. 2). This difficulty can be overcome by the application of damage assessment rules such as Miner's rule.

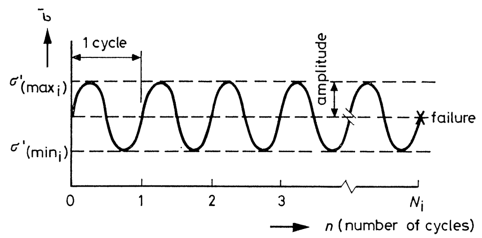


Fig. 1. Constant-amplitude test.

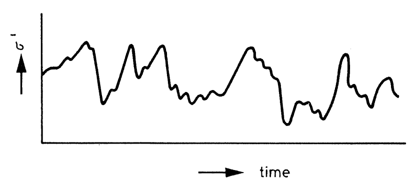


Fig. 2. Erratically varying stress.

It would be outside the scope of this paper to deal more fully with the stochastic character of the loading. In principle, it is presupposed that the statistical properties of the loading or stress are sufficiently known to enable its erratic time-dependent behaviour to be described accurately enough for the purpose.

## 2 Damage

In order to account for the phenomenon of fatigue it is assumed that with every change in stress a certain amount of damage occurs in the concrete. The overall damage that arises as the result of a number of stress cycles may result in a lowering of the strength.

Hence an obvious approach is to choose units of damage in the material, i.e. in the concrete, as the starting point for the assessment of the fatigue effect in consequence of erratic stress variations (with a non-constant-amplitude, mean value and/or frequency). Such assessment is necessary because a great many types of such stresses are possible.

Although micromechanical investigation of materials has revealed that the damage that occurs is associated with the formation of cracks and with changes in the length, width and direction of the cracks, it has hitherto not proved possible to derive from such research a serviceable method of measurement for ascertaining the degree of damage.

Instead of basing oneself on a damage unit capable of physical interpretation it is also possible to adopt hypothetical units. In that case it will, however, be necessary to verify the hypothesis.

## 3 Miner's rule

The most commonly employed hypothesis for determining the degree of damage due to erratically varying stresses is Miner's hypothesis [1]. He used it in 1945 to interpret the results of tests on aluminium alloy bars using a simple loading program comprising two blocks. It has become common practice to apply this hypothesis also to concrete structures and also in cases where the stresses display a more erratic time-dependent behaviour. Indeed, it is now so widely accepted that it is referred to as "Miner's rule". For applications of this kind, however, it has hardly been verified as yet. IBBC-TNO (Institute TNO for Building Materials and Building Structures) has, in collaboration with Stupoc (Steering Group on Offshore Structure Problems) and CUR (Netherlands Committee for Concrete Research), made a start with this verifi-

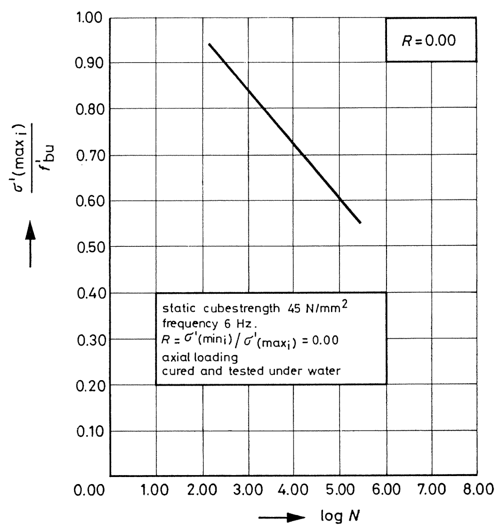


Fig. 3. Wöhler-diagram.

cation for normalweight concrete subjected to compressive stresses due to axial load.

Miner's rule establishes a relationship between the damage due to erratically varying stresses and the results of constant-amplitude tests. In such tests the number of stress cycles $N_i$ needed to produce failure is determined (Fig. 1).

The results of a number of these tests performed at various levels of stress are represented as so-called Wöhler lines. For concrete the usual practice is to indicate the value of $N_i$ on a logarithmic scale and the value of the applied maximum stress on a linear scale. Plotted in this way, the results will conform approximately to a straight line. A line of this kind is presented in Fig. 3, which is based on results of research conducted by IBBC-TNO [2]. In that research it was found, on the basis of some 200 constant-amplitude tests, that the location of a Wöhler line is dependent on a large number of factors such as the minimum stress, the frequency of the cycles, the grade of the concrete, the degree to which the concrete has achieved full hardening, and the conditions of curing and testing.

Miner's rule is based on the conception that the contribution to damage in consequence of a single stress cycle ranging from a maximum stress $\sigma'(\max_i)$ to a minimum stress $\sigma'(\min_i)$ has a magnitude $1/N_i$, where $N_i$ denotes the number of stress cycles which results in failure in a constant-amplitude test at the same level of stress, provided that the conditions pertaining to the single cycle are the same as those partaining to the constant-amplitude test. Therefore, besides taking account of the level of the stress cycle, Miner's rule also takes account of the factors already referred to earlier on, such as the frequency, the quality of the concrete, the degree of hardening and the conditions of testing and curing.

The damage due to a number of stress cycles $c$ will, according to Miner's rule, make a contribution $M$ which is the sum of the damage contributions of each of the individual cycles. Stated as a formula, this rule is:

$$M = \sum_{i=1}^{c} \frac{1}{N_i} \qquad (1)$$

The rule is moreover linked to a failure criterion in that it is presupposed that failure occurs when $M$ becomes equal to unity (i.e., when $M = 1$). In the following $M$ will be called the Miner sum, while the value corresponding to the limit of failure is called the Miner number.

Apart from any safety margins that may be introduced, this conception leads to the following formulation of the limit state with regard to fatigue:

$$M = \sum_{i=1}^{c} \frac{1}{N_i} \leqq 1 \qquad (2)$$

The value 'one' for the Miner number is deduced directly from a constant-amplitude test. If Miner's rule is to have general validity, it will also have to be valid for a test of that kind, in which case the number of cycles $c$ is equal to $N_i$, so that $M$ becomes equal to $N_i/N_i = 1$.

For concrete loaded in compression, however, the assumption of a Miner number $M = 1$ on the basis of constant-amplitude test results is not correct. According to [2] it emerges that $N_i$ is a stochastic variable with a logarithmic-normal distribution. (In this paper these tests are comprised in the test series A and G). It can be shown that the Miner number must also be a stochastic variable with a likewise approximately logarithmic-normal distribution (see Appendix 1). Basing oneself on the result of a constant-amplitude test, it can be shown (see Appendix II) that the median of the Miner number and the standard deviation of the logarithmic value of the Miner number are equal to:

$$m(M) = 10^{1,15 s^2(\log N_i)} \tag{3}$$

$$s(\log M) = s(\log N_i) \tag{4}$$

where $s(\log N_i)$ is the standard deviation of the logarithm of the number of stress cycles $N_i$ up to failure in a constant-amplitude test.

As the Miner number $M$ has a logarithmic-normal distribution it would be obvious to look at the logarithmic value of the Miner number $\log M$, because this value has a normal distribution. This distribution is more familiar and gives easier insight than the logarithmic-normal distribution.

Nevertheless in equation (3) the median (50% probability) of the Miner number $M$ is given. (This median is in fact the Miner number corresponding with the mean-value of $\log M$). This procedure is choosen because of the fact that the Miner number is directly related to the life time of the structure.

It appears therefore that the dispersion in a constant-amplitude test is of fundamental importance with regard to the probability of the occurence of a particular Miner number i.e., the probability density function. Since there undoubtedly exists a relation between the dispersion in the static compressive strength of concrete and the dispersion $s(\log N_i)$ in a constant-amplitude test, it can be presumed that the dispersion in the static compressive strength has a share in determining the probability density function of the Miner number.

Hence it follows that for an experimental verification of Miner's rule it is necessary to pay attention to the relation between the static compressive strength, the number of load cycles $N_i$ up to failure in a constant-amplitude test, and the value of the Miner number. In such a verification the effect of dispersion should be considered more particularly.

## 4 Experimental research

### 4.1 *General*

Fatigue testing was done by means of tests performed on plain concrete. The specimens were subjected to axial compressive load. In order to obtain a homogeneous stress in them, cylindrical specimens were used. These were 450 mm high and 150 mm in diameter and consisted of concrete made with gravel aggregate of up to 32 mm particle size. The end faces of the cylindres were ground flat.

Two types of concrete were used: a high strength concrete with a cube strength at 28 days of 45 N/mm$^2$ (coefficient of variation 5%) and a lower-strength concrete with a cube strength at 28 days of 30 N/mm$^2$ (coefficient of variation 10%). The composition of the two types of concrete is indicated in table 1.

Table 1. Composition of the concrete.

| | high-strength concrete | low-strength concrete |
|---|---|---|
| portland cement, ordinary | 360 kg | 284 kg |
| water | 162 kg | 170 kg |
| water cement ratio | 0.45 | 0.60 |
| gravel | 1860 kg | 1871 kg |
| fineness modulus | 5.40 | 5.40 |

The test set-up with the servo-hydraulic control equipment is illustrated in Fig. 4. The tests were in all cases performed with a constant frequency.

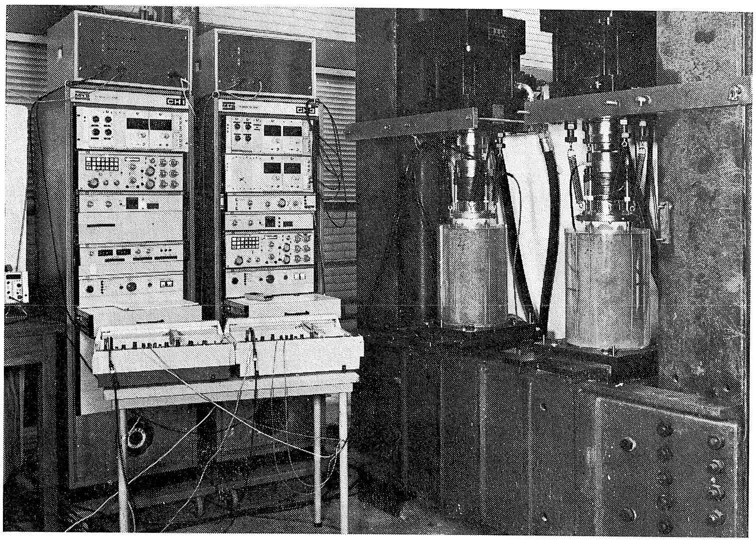

Fig. 4. Test set-up with servo-hydraulic control equipment.

## 4.2 *Constant-amplitude tests: series A to G*

The constant-amplitude testing program (see table 2) had two objectives. First was to determine by means of test series A and G values of $N_i$ which were used in the non-constant-amplitude tests for calculating the Miner sum. The other objective was to measure the influence of a number of parameters. For this purpose the following were varied in the test series A to G:

- the curing and testing conditions; "wet" means cured and tested under water; "dry" means tested and cured at 20 °C and 50–65% R.H.;
- the age of the concrete at the time of testing; usually testing started at an age of 28 days; two series were, however, tested at an age of $\frac{1}{2}$ and 1 year;
- the frequency; to speed up the time of testing and to avoid further hardening of the concrete most of the tests were performed at a frequency of 6 Hz; to gain some insight into the influence of the frequency, one series was tested at a frequency of 0.7 Hz;
- the concrete quality (strength class).

Table 2. Program of constant-amplitude tests.

| series | concrete quality at 28 days | curing and testing conditions | age | frequency | number of tests |
|---|---|---|---|---|---|
| A | B 45 | wet | 28 days | 6 Hz | 60 |
| B | B 45 | dry | 28 days | 6 Hz | 21 |
| C | B 30 | wet | 28 days | 6 Hz | 38 |
| D | B 45 | wet | $\frac{1}{2}$ year | 6 Hz | 38 |
| E | B 45 | wet | 1 year | 6 Hz | 46 |
| F | B 45 | wet | 28 days | 0,7 Hz | 41 |
| G | B 45 | wet | 28 days | 6 Hz | 86 |
| | | | | | 330 |

The tests were performed at eleven different levels of the maximum stress $\sigma'_{max}$ in relation to the static strength $f'_{bu}$ of the cilinders tested.

The maximum stress levels were combined with five stress ratios $R = \sigma'_{min}/\sigma'_{max}$.

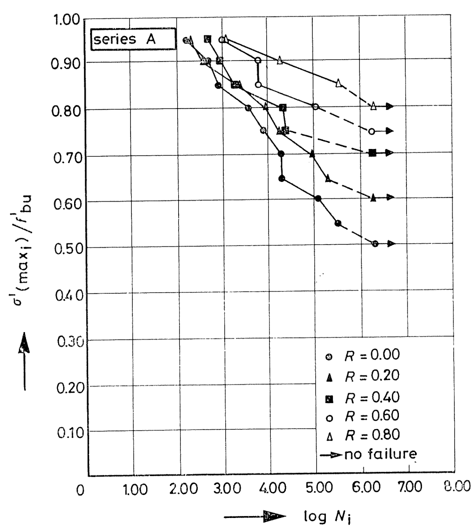


Fig. 5. Wöhler curves of test series A.

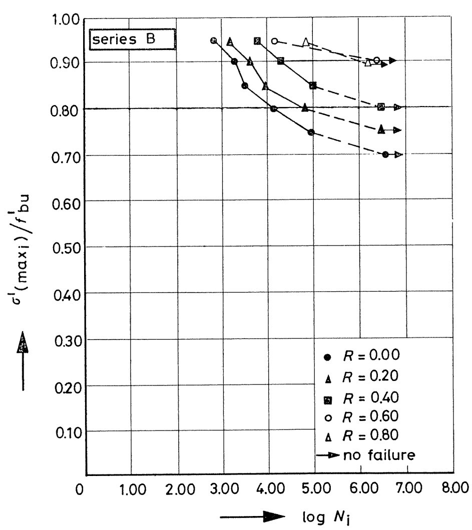


Fig. 6. Wöhler curves of test series B.

The maximum stress varied from $0.45 f'_{bu}$ to $0.95 f'_{bu}$ and the stress ratio from 0 to 0.8.

The test results of the series A to F are given in Fig. 5 to 10 in the form of Wöhler curves. If at a certain stress level more than one value of $N_i$ was available, than the mean value of log $N_i$ is given.

From series A the test with $\sigma'_{\max} = 0.70 f'_{bu}$ and $R = 0$ was repeated 21 times. The results of that tests are given in a relative cumulative frequency diagram in Fig. 11.

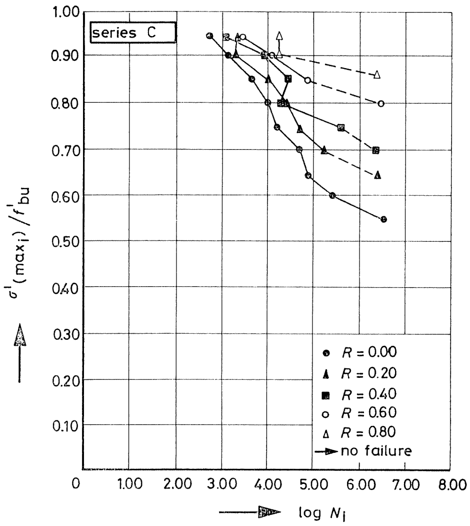


Fig. 7. Wöhler curves of test series C.

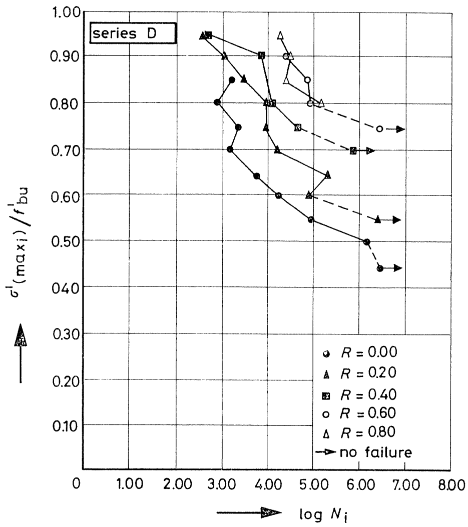


Fig. 8. Wöhler curves of test series D.

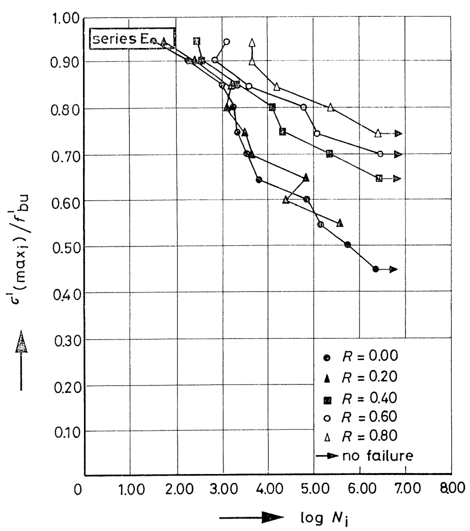


Fig. 9. Wöhler curves of test series E.

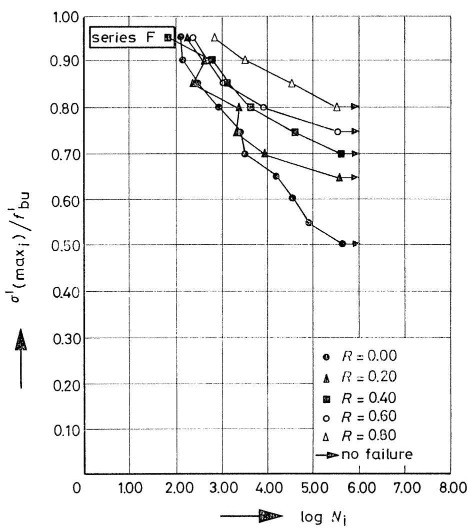


Fig. 10. Wöhler curves of test series F.

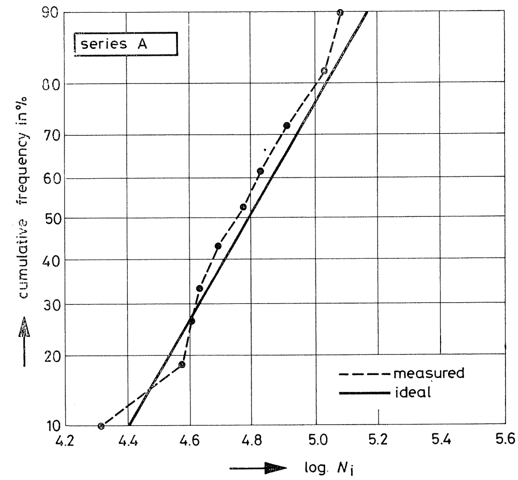


Fig. 11. Cumulative frequency diagram for 21 alike tests from test series A.

In Figs. 12 tot 14 the results of test G are given in the form of Wöhler curves. As in this series each test was repeated at least seven times, it was possible to give curves for the mean value of log $N_i$ and to indicate the dispersion by means of curves at an distance corresponding to the standard deviation $s(\log N_i)$. For comparison lines at a distance corresponding to the standard deviation of the relative static strength $s(f'_{bu}) = 0.03$ are also given.

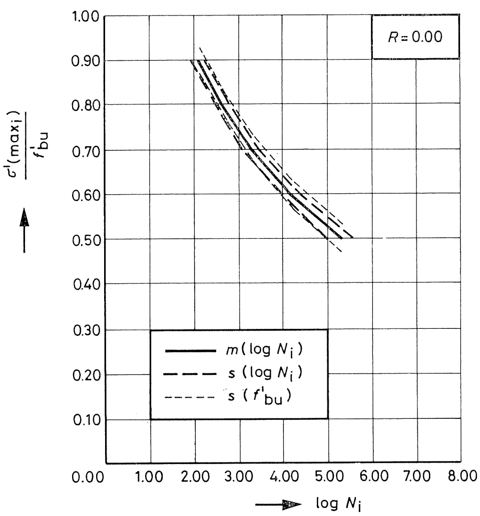


Fig. 12. Wöhler curves of test serie G. for R = 0.00

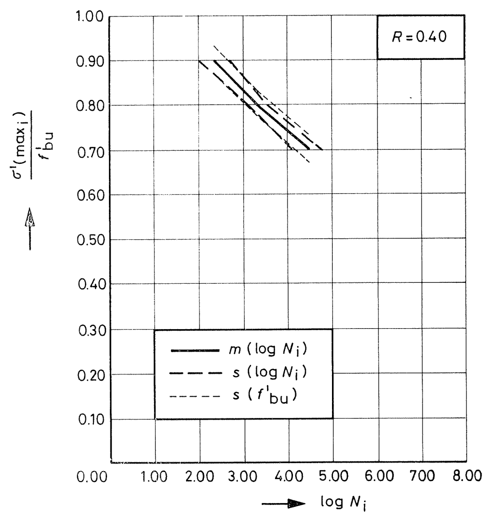


Fig. 13. Wöhler curves of test serie G. for R = 0.40

A number of conclusions can be drawn from the tests:

- concrete with a higher static compressive strength is, for equal stress limits, able to endure more stress cycles up to failure (i.e. possesses greater fatigue strength or a longer working life) irrespective of whether the higher strength is obtained by better concrete mix composition (see series A and C) or as a result of greater age (see series A, D and E); the increase in fatigue strength is, however, less than proportional to the increase in static strength;
- concrete which has been cured and tested under water is found to have a higher static compressive strength than concrete of the same composition which has been cured and tested under dry conditions (see series A and B); the fatigue strength is, however, practically the same in both cases;
- the frequency of the stress cycles is found to have a distinct effect on the fatigue strength, loading applied at the higher frequency is less detrimental than at the lower frequency (see series A and F);
- from the 21 similar tests of series A and from series G is follows that the dispersion in $N_i$ can, with fair approximation, be represented by a logarithmic-normal distribution (see Fig. 11);
- the magnitude of the dispersion in log $N_i$ is chiefly dependent on the value of the stress ratio $R$ (see Figs. 12 to 14);
- the dispersion in log $N_i$ has the same magnitude as can be expected in view of the dispersion in the static compressive strength.

It should be noted that, although the concrete quality and the conditions of testing were the same for series A and G, relatively large differences in their respective test

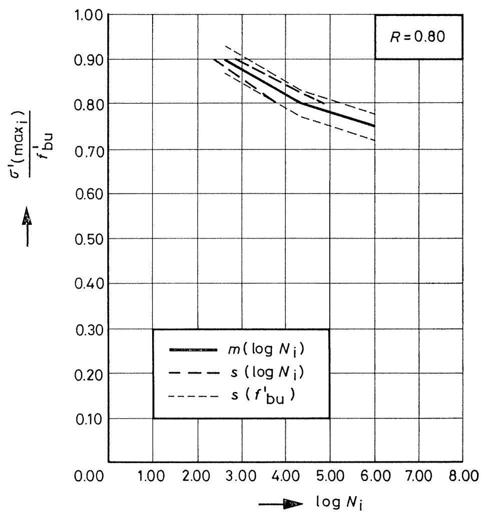


Fig. 14. Wöhler curves of test serie G. for $R = 0.80$

results nevertheless occurred. A conclusive explanation for this has not yet been found.

### 4.3 *Program loading tests: series H, I and J*

In a 'program loading test' a test specimen is loaded with a number of successive stress blocks. A "block" is characterized by $\sigma'(\max_i)$, a certain stress ratio $R = \sigma'(\min_i)/\sigma'(\max_i)$ and a certain number of cylces $n_i$ (see Fig. 15). In this section of

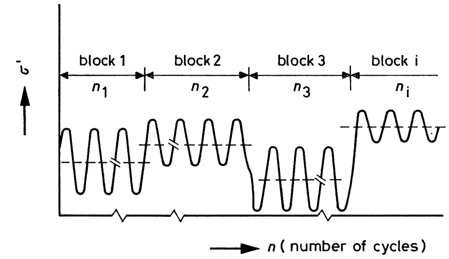


Fig. 15. Program loading test.

the research project, which comprised about 100 specimens, three different test series were performed, namely:

*Series H*: the loading program comprised only two blocks. After the predetermined number of cycles had been completed in the first block, the program changed over to the second (and last) block, in which the number of cycles up to failure was determined; although in this case the tests were not of constant amplitude, it must be admitted that an important aspect of the erratic stress behaviour in relation to time, namely the random sequence of the stress variations, was largely absent in this series; this drawback is less pronounced in series I and J.

*Series I*: the loading program comprised sequences of two, four or eight blocks; after a sequence had been completed, it was repeated until failure occurred; these sequences differed from one another not only in the number of blocks, but also in the order in which the blocks occurred in the sequence: sometimes the test was started with smaller, sometimes with larger amplitudes; sequences in which the blocks were applied in a more random succession were also performed.

*Series J*: the testing procedure was similar to that of series I, except that the average stress per block was also varied in a test.

The Miner numbers were calculated on basis of the results of test series A.

The results of these test series are presented in the form of cumulative frequency distributions in Figs. 16, 17 and 18. The values of log $M$ have been plotted on the horizontal axis, while the cumulative frequency has been plotted on the vertical axis to a scale with normal distribution. With a scale of this kind, a linear relation between log $M$ and the frequency of occurrence should be found if the Miner number conforms to a logarithmic-normal distribution.

In the special case of series H two distributions are presented in Fig. 16. One of them relates to all the test results, whereas in the other distribution the results of those tests in which the specimens failed in the first block have been omitted, since these were in fact constant-amplitude tests.

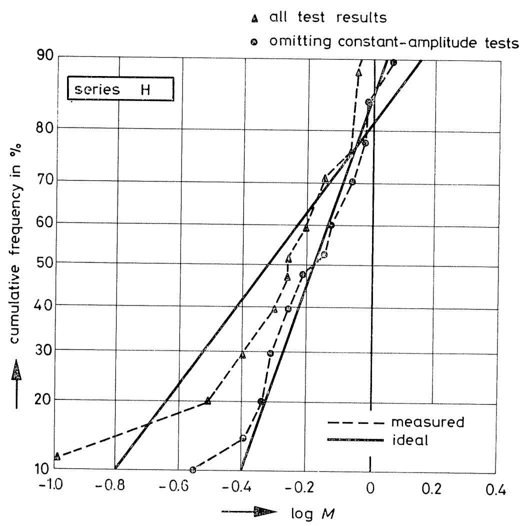


Fig. 16. Cumulative frequency diagram for test series H.

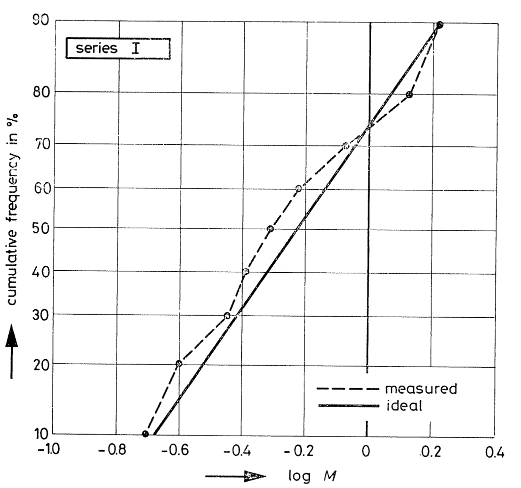


Fig. 17. Cumulative frequency diagram for test series I.

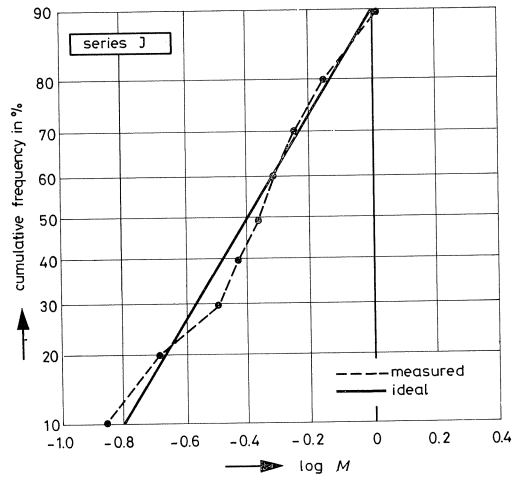


Fig. 18. Cumulative frequency diagram for test series J.

The combined result of all the program loading tests is given in Fig. 19. For the individual test series H, I and J, and for these series combined, the mean value and the standard deviation of log $M$ as well as the median of $M$ are given in table 3.

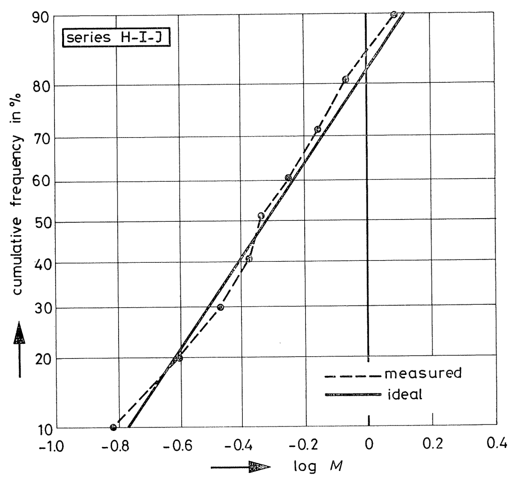


Fig. 19. Cumulative frequency diagram for test series H, I and J.

Table 3. Results of program loading tests, series H, I and J.

| series | $m(\log M)$ | $m(M)$ | $s(\log M)$ |
|---|---|---|---|
| H (all tests) | −0.311 | 0.49 | 0.369 |
| H (with omissions) | −0.192 | 0.64 | 0.197 |
| I | −0.225 | 0.60 | 0.347 |
| J | −0.411 | 0.39 | 0.308 |
| H, I, J | −0.328 | 0.47 | 0.346 |

The following conclusions can be drawn from the results of these tests:

- the value of the Miner number can be described with a logarithmic-normal distribution;
- the median of the logarithmic-normal distribution of $M$ is lower than could be expected on the basis of constant-amplitude tests; from the results of the 21 similar tests of test series A, namely, $s(\log N_i) = 0.298$, a median:

$$m(M) = 10^{1.15 s^2(\log N_i)} = 10^{1.15 . 0.298^2} = 1.27$$

  was expected according to equation (3), whereas the measured value is even less than unity;
- the dispersion in log $M$ per test series is more or less constant; an exception to this is series H, in which part of the results have been omitted and which shows less dispersion, which is understandable because a number of "maverick" values have not been included in the results;
- $s(\log M)$ has approximately the value predicted by formula (4): 0.298.

### 4.4 *Variable-amplitude tests: series K to O*

In variable-amplitude tests the value of the amplitude changes in each cycle or half cycle. The cycles themselves remain sinusoidally. With such tests it is, even more than with program loading tests, possible to achieve a random sequence of cycles.

In this research several types of variable-amplitude tests were performed, namely:

series K, comprising tests with a stationay mean stress (Fig. 20a);
series L and M, comprising tests with a stationary minimum stress (Fig. 20b);
series M and O, comprising tests with a stationary maximum stress (Fig. 20c).

Table 4. Test program and results of series K.

| test program | | | results | | |
|---|---|---|---|---|---|
| mean $m(\sigma) \times f'_{bu}$ | standard deviation $s(\hat{\sigma}) \times f'_{bu}$ | dispersion range $w \times f'_{bu}$ | $m(\log M)$ | $m(M)$ | $s(\log M)$ |
| 0.65 | 0.0825 | 0.50 | −0.278 | 0.53 | 0.439 |
| 0.65 | 0.125 | 0.50 | −0.193 | 0.64 | 0.235 |
| 0.65 | 0.166 | 0.50 | −0.338 | 0.46 | 0.197 |
| 0.575 | 0.0825 | 0.50 | −0.204 | 0.63 | 0.339 |
| 0.575 | 0.125 | 0.50 | −0.322 | 0.48 | 0.431 |
| 0.575 | 0.166 | 0.50 | −0.307 | 0.49 | 0.446 |
| 0.50 | 0.0825 | 0.50 | 0.249* | 1.77 | 0.713* |
| 0.50 | 0.125 | 0.50 | −0.126 | 0.75 | 0.406 |
| 0.50 | 0.166 | 0.50 | 0.077 | 1.19 | 0.366 |
| 0.575 | 0.125 | 0.60 | −0.071 | 0.85 | 0.479 |
| 0.575 | 0.166 | 0.60 | −0.148 | 0.71 | 0.302 |
| 0.50 | 0.125 | 0.60 | −0.209 | 0.62 | 0.257 |
| 0.50 | 0.166 | 0.60 | −0.060 | 0.87 | 0.399 |

* A few tests were terminated before the occurrence of failure

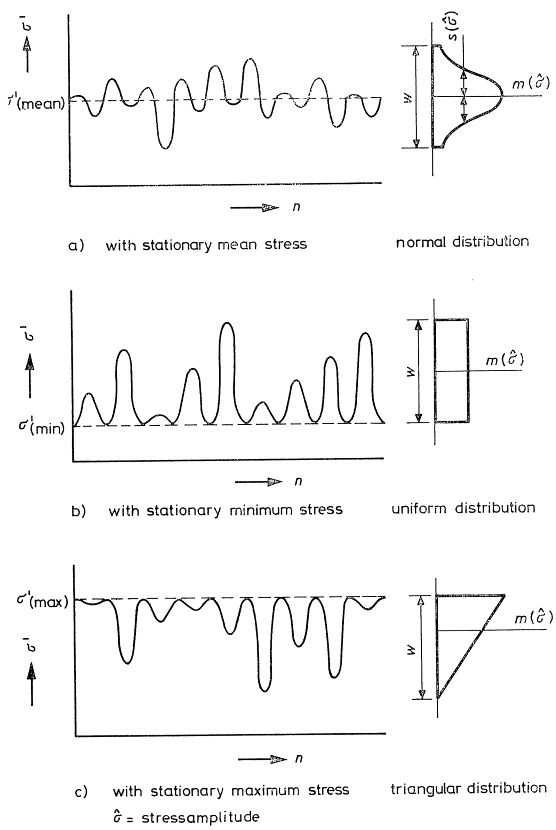


Fig. 20. Variable amplitude tests.

In all these tests the frequency was kept constant (6 Hz).
The variation of the amplitude values was described with probability density functions (see Fig. 20). The sequence of the cycles was, except for series N and O, of a random character. Nevertheless the tests performed are completely reproducible because a digital variable-amplitude generator specially developed by the investigating laboratory was used for the purpose [3]. The sequence of the cycles in series N and O was regular.

In the variable-amplitude tests in series *K* the amplitude values were always normally distributed (see also Fig. 20a). For the program of tests in question table 4 gives the mean stress $m(\sigma)$, the standard deviation $s(\hat{\sigma})$ of the amplitudes, and the dispersion range $w$. The results of the tests are given in the form of the mean $m(log\ M)$ of the logarithm of the Miner number the median $m(M)$ and the standard deviation $s(\log M)$. Each test was repeated at least seven times, and 96 test were performed in all.

For calculating the Miner number the median $m(N_i)$ was derived from series *G*. Further it was assumed that the damage caused by a half cycle is equal to the half of the damage caused by a full cycle. This assumption seems to be reasonable because the probability density functions of the amplitudes were symmetric.

From all the test results together it emerges that $m(\log M) = -0.150$ – hence $m(M) = 0.71$ – and $s(\log M) = 0.416$. In Fig. 21 the results are presented in the form of a cumulative frequency diagram. For comparison, the distribution obtained from test series H, I and J (see also Fig. 19) has been included.

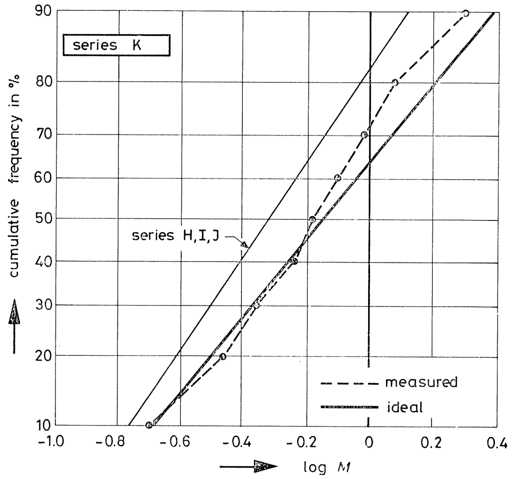


Fig. 21. Cumulative frequency diagram for test series K.

In the variable-amplitude tests with stationary minimum stress (series L) and with stationary maximum stress (series M) various forms of probability density functions were employed, namely, the normal distribution, the uniform distribution and the triangular distribution with the highest frequency either at the lowest or at the highest amplitude value. Each test was repeated 7 times.

In all, 56 tests were carried out. The test program and results are given in table 5. The dispersion range $w$ was always $0.5f'_{bu}$.

For the tests of series L in which the minimum stress was stationary it was found that $m(\log M) = 0.148$ – hence $m(M) = 1.41$ – and $s(\log M) = 0.365$, while for those of series $M$ these values were $m(\log M) = -0.641$ – hence $m(M) = 0.23$ – and $s(\log M) = 0.368$. In Fig. 22 the results are presented in the form of cumulative frequency diagrams.

It can be concluded from the series L and M that variable-amplitude tests at a relatively high stress level (series L) are more damaging than at a relatively low stress level (series M). To find out if the random sequence of the amplitudes or the level of the stationary stress dominates the extra damage the tests with the uniform distribution were repeated on the understanding that the sequence of the amplitudes was regular. Half of the tests started with an amplitude of zero; in 4096 (= 4 K)

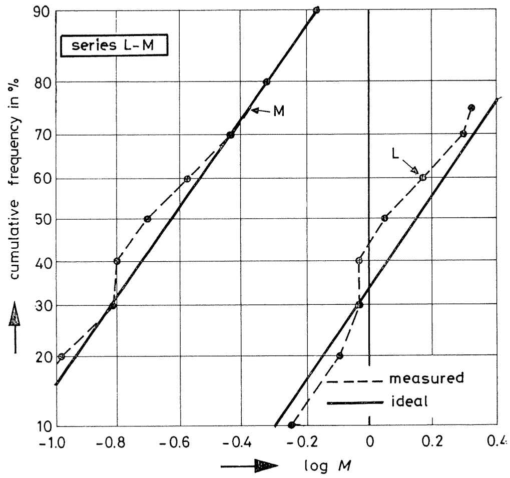


Fig. 22. Cumulative frequency diagram for test series L and M.

Table 5. Test program and results of series L and M.

| series | test program | | | | results | | |
|---|---|---|---|---|---|---|---|
| | stationary stress $\times f'_{bu}$ | type of distribution | mean $m(\hat{\sigma}) \times f'_{bu}$ | standard deviation $s(\hat{\sigma}) \times f'_{bu}$ | $m(\log M)$ | $m(M)$ | $s(\log M)$ |
| L | $\sigma'_{min} = 0.25$ | normal | 0.5 | 0.125 | 0.304 | 2.01 | 0.478 |
| | $\sigma'_{min} = 0.25$ | uniform | 0.4 | – | 0.011 | 1.03 | 0.195 |
| | $\sigma'_{min} = 0.25$ | triangular | 0.583 | – | 0.156 | 1.43 | 0.182 |
| | $\sigma'_{min} = 0.25$ | triangular | 0.417 | – | 0.122 | 1.32 | 0.497 |
| | | | | together | 0.148 | 1.41 | 0.365 |
| M | $\sigma'_{max} = 0.75$ | normal | 0.5 | 0.125 | −0.818 | 0.15 | 0.460 |
| | $\sigma'_{max} = 0.75$ | uniform | 0.5 | – | −0.653 | 0.22 | 0.226 |
| | $\sigma'_{max} = 0.75$ | triangular | 0.417 | – | −0.379 | 0.42 | 0.393 |
| | $\sigma'_{max} = 0.75$ | triangular | 0.583 | – | −0.715 | 0.19 | 0.268 |
| | | | | together | −0.641 | 0.23 | 0.368 |

cycles the amplitude increased gradually to the maximum value and in the next 4096 cycles it decreased to zero. This sequence was repeated until failure occurred.

The other half of the tests were started with the maximum aplitude. Each sequence was combinated with either a stationary minimum stress of $0.25f'_{bu}$ or a stationary maximum stress of $0.75f'_{bu}$. Each combination was repeated 7 times. In all, 28 tests were carried out. The test program and the results are given in table 6.

In Fig. 23 the results of all the tests of series N and O are presented in the form of cumulative frequency diagram. For comparision, the results of the tests with the uniform distribution of the series L and M are also given.

As can be seen from the test results, the regular sequence of the amplitudes is

Table 6. Test program and results of series N and O.

| | test program | | | results | | | |
|---|---|---|---|---|---|---|---|
| series | stationary stress $\times f'_{bu}$ | type of distribution | mean $m(\hat{\sigma}) \times f'_{bu}$ | started with amplitude | $m(\log M)$ | $m(M)$ | $s(\log M)$ |
| N | $\sigma'_{min} = 0.25$ | uniform | 0.5 | zero | −0.100 | 0.79 | 0.210 |
| | | | | maximum | −0.194 | 0.64 | 0.254 |
| | | | | together | −0.147 | 0.71 | 0.229 |
| O | $\sigma'_{max} = 0.75$ | uniform | 0.5 | zero | −0.280 | 0.52 | 0.255 |
| | | | | maximum | −0.568 | 0.27 | 0.238 |
| | | | | together | −0.424 | 0.38 | 0.320 |

more damaging with stationary minimum stresses (series N versus series L) and less damaging with stationary maximum stresses (series O versus series M) than is the random sequence. Variable-amplitude tests with a random sequence seem to give extreme values for the Miner number.

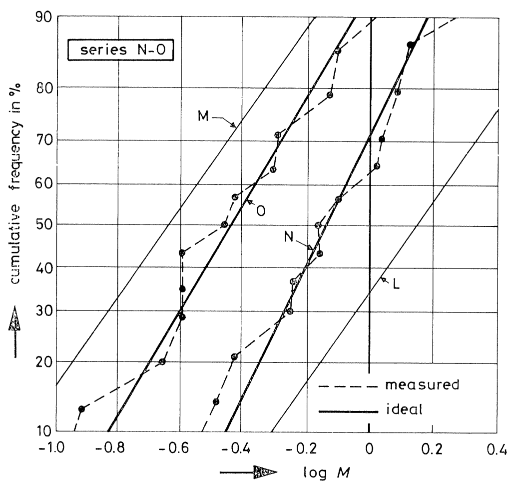


Fig. 23. Cumulative frequency diagram for test series N and O.

## 5 Interpretation

From the program loading tests as well as the variable-amplitude tests it emerges that the value of the Miner number can be satisfactorily described with a logarithmic-normal distribution. This was also to be expected on the ground of theoretical considerations (see Chapter) 3).

By virtue of the logarithmic-normal distribution of $N_i$ and in accordance with the formulas (3) and (4) the median value and the standard deviation of the Miner num-

Table 7. Comparison of theoretical and measured results of non-constant amplitude tests.

| series | $s(\log N_i)$ | from series | $m(M)$ measured | $s(\log M)$ measured | $m(M)$ from (3) | $s(\log M)$ from (4) |
|---|---|---|---|---|---|---|
| H | 0.298 | A | 0.49 | 0.369 | 1.27 | 0.298 |
| I | 0.298 | A | 0.60 | 0.347 | 1.27 | 0.298 |
| J | 0.298 | A | 0.39 | 0.308 | 1.27 | 0.298 |
| K | 0.142–0.538 | G | 0.71 | 0.416 | 1.05–2.15 | 0.142–0.538 |
| L | 0.142–0.538 | G | 1.41 | 0.365 | 1.05–2.15 | 0.142–0.538 |
| M | 0.142–0.538 | G | 0.23 | 0.368 | 1.05–2.15 | 0.142–0.538 |
| N | 0.142–0.538 | G | 0.71 | 0.229 | 1.05–2.15 | 0.142–0.538 |
| O | 0.142–0.538 | G | 0.38 | 0.320 | 1.05–2.15 | 0.142–0.538 |
| WEIGLER [4] | ≈1.05 | | ≈2.0 | ≈0.9 | 18 | 1.05 |
| WEIGLER [5] | ≈0.95 | | ≈1.0 | ≈0.9 | 7 | 0.95 |
| TEPFER [6] | ≈1.15 | | ≈0.63 | ≈0.95 | 33 | 1.15 |

ber can be calculated. For the various test series table 7 indicates which values of $m(M)$ and $s(\log M)$ were measured and which were expected on the basis of these formulas.

The table also includes the results of program loading tests of WEIGLER [4 and 5] and TEPFERS [6]. As these results were, generally speaking, not available in the form used in this resport, they have been obtained by measurement from diagrams, etc., so that there may be minor deviations from the actual results obtained by those investigators.

It appears from the table that in general the measured median value $m(M)$ is smaller then the median value predicted by the formula. This indicates that in non-constant-amplitude tests a higher degree of damage occurs than in constant-amplitude tests.

The difference between the values $m(M)$ of series L, M, N and O indicates that this higher degree of damage is related to the type of loading program, and the sequence of the cycles.

The standard deviation $s(\log M)$ that has been measured both in the program loading tests and the variable-amplitude tests, has always the magnitude that could be expected according to formula (4).

From the test results it can be concluded that:

- the measured distribution of the Miner number is in accordance with the distribution which could be expected on the ground of theoretical considerations;
- the standard deviation of the Miner number $s(\log M)$ is almost equal to the value $s(\log N_i)$ that could be expected on the ground of the theoretical considerations mentionated;
- the standard deviation $s(\log N_i)$ is of the magnitude that could be predicted on the basis of $s(f'_{bu})$;
- in general more damage appears in non-constant-amplitude tests than could be predicted on the basis of constant-amplitude tests in combination with Miner's rule.
- the extra damage is related to the stress level and the sequence of the cycles.

From table 7 it follows that the test results of both WEIGLER and TEPFERS are in accordance with the conclusions mentioned before. This is so in spite of the relatively large scatter in their investigations.

In view of this result it seems to be justified to use Miner's rule to take account of the effect of erratically varying stresses. This conclusion is however only valid within the restrictions of the testing program, like the type of loading, the frequency, the concrete quality etc.

## 6 Semi-probabilistic analysis

The safety of a structure manifests itself in the margin that exists between the effect of the loading and the resistance of the structure. In the case of fatigue the safety can be defined in two ways, namely:

- as the margin between the magnitude of the existing load and the strength of the structure;
- as the margin between the intended lifetime and the expected lifetime of the structure.

The difference between these two definitions is due to the fact that in the material both the level of the stress cycles and the number of cycles are determining factors with regard to failure.

On the basis of the results and the experience gained from the research described in the foregoing, it is possible to indicate how a safety margin with regard to fatigue failure can be defined with the help of Miner's rule and the distinction between Miner sum and Miner number. For this purpose the starting point adopted is the semi-probabilistic design procedure (level I procedure) as envisaged, inter alia, in the CEB-FIP Model Code for Concrete Structure. In accordance with that Code it must be shown, in the ultimate limit state, that the effect $S_d$ of the characteristic load $F_k$ multiplied by a load factor $\gamma_f$ (the so-called design load $F_d$) is less than the resistance $R_d$ of the structure. This resistance $R_d$ is dependent on the design strength $f_d$ of the material, which is the characteristic material strength $f_k$ divided by a material factor $\gamma_m$. Hence

$$S_d(F_k \cdot \gamma_f) \leqq R_d(f_k/\gamma_m) \tag{5}$$

In the case of cyclic loading the effect thereof is the occurrence of damage in the material. This research has shown that Miner's rule can be used for determining the degree of damage, so that the effect of the load $S_d(F_k \cdot \gamma_f)$ thus becomes the Miner sum $M_{Fd}(F_k \cdot \gamma_f)$, which can be determined as follows:

$$M_{Fd} = M_{Fd}(F_k \cdot \gamma_f) = \sum_{i=1}^{c} \frac{1}{N_i} \tag{6}$$

In this expression $c$ denotes the number of stress cycles which occurs in the intended lifetime of the structure in consequence of the characteristic loading (defined as a conservative load-time history) multiplied by $\gamma_f$. A complication in determining the Miner sum is due to the fact that it is necessary to make use of a stochastic material property, namely, the number of constant-amplitude stress cycles $N_i$ up to failure. For $N_i$ the median value $m(N)$ has to be taken so that no safety margin with regard to a material strength is introduced, because according to formula (5) $\gamma_m$ can only be introduced in $R_d$. Equation (6) then becomes:

$$M_{Fd} = M_{Fd}(F_k \cdot \gamma_f) = \sum_{i=1}^{c} \frac{1}{10^{m(\log N_i)}} = \sum_{i=1}^{c} \frac{1}{m(N)} \tag{7}$$

It is to be noted that the value of $\gamma_f$ is related to the load factor that should be applied in a corresponding analysis with regard to (quasi) static loads. This factor $\gamma_f$ serves to take account of a number of similar uncertainties.

The resistance $R_d$ of the structure can be expressed in a Miner number $M_{fd}$. As appears from the research, the Miner number can be conceived as a stochastic variable characterized by a logarithmic-normal distribution. If the values of $m(\log M)$ and $s(\log M)$ are known, it is possible to determine a characteristic value of the Miner number according to:

$$M_{fk} = 10^{[m(\log M) - \beta \cdot s(\log M)]} \tag{8}$$

The value of $\beta$ depends on the accepted probability of $M_{fk}$ not being reached.

Next, the design value $M_{fd}$ must be determined with the aif of $M_{fk}$ and $\gamma_m$. Having regard to the logarithmic-normal character of the Miner number, the following obvious definition suggests itself for $M_{fd}$:

$$M_{fd} = 10^{[m(\log M) - \beta \cdot s(\log M)]/\gamma_m} \tag{9}$$

The probability of the occurrence of $M_{fd}$ is thus defined similarly to the design strenght in the case of (quasi) static load.

For simplification a value $\beta^* (\beta^* > \beta)$ can be so introduced into (9) that the design value $M_{fd}$ is directly obtained. The procedure for the analysis can now, starting from the equations (5), (7) and (9), be summarized as follows:

$$M_{Fd}(F_k \cdot \gamma_f) \leqq M_{fd}$$

$$\sum_{i=1}^{c} \frac{1}{m(N)} \leqq 10^{[m(\log M) - \beta^* \cdot s(\log M)]} \tag{10}$$

The procedure outlined above is so contrived that the value for $M_{fd}$ must be determined by means of material research with appropriate stress-time relations. It is then up to the designer to calculate the Miner sum $M_{Fd}$. This operation is reduced to the level of constant-amplitude stresses. Non-constant-amplitude stresses will therefore have to be reduced, with the aid of a suitable counting method, to the level of constant-amplitude stresses.

## 7 Fatigue limit Series P

Usually constant amplitude tests are performed up to about $2 \times 10^6$ cycles. This limit was also used in this investigation because of the fact that with higher limits testing takes too much time. This is expensive, and in consequence of the further hardening of the concrete the test results will be too optimistic.

For calculating the Miner sum it can nevertheless be important to know what damage is caused by cycles with low stress levels.

In the literature a fatigue limit is often adopted. This means that there is a maximum stress level below which no fatigue damage occurs even with very high numbers of cycles. For this fatigue limit a maximum stress of about $0.5f'_{bu}$ is often mentioned. Experimental evidence for this fatigue limit is never given, however.

As a first step in establishing evidence for the existence of a fatigue limit, in series P 54 similar constant-amplitude tests were performed. In these tests the stress varied from a minimum of zero to a maximum of $0.4 \times f'_{bu}$.

Apart from the stress level, the same conditions were maintained in this series as in series G. Testing was continued up to at least $2 \times 10^6$ cycles unless failure occurred.

The result of the testing was that two out of 54 specimens failed before $2 \times 10^6$ cycles were reached. Another result was that in the tests which were not stopped at $2 \times 10^6$ cycles three other specimens failed.

Failure occurred at the following numbers of cycles:

$N_1 = 1.782.710$
$N_2 = 1.914.170$
$N_3 = 2.830.130$
$N_4 = 3.173.210$
$N_5 = 3.266.710$

From these test series it can be concluded that

- at $0.4f'_{bu}$ no fatigue limit exists;
- after more than $2 \times 10^6$ cycles failure is possible.

## 8 Further research

To provide a firmer base for the analysis procedure envisaged in equation (10) it would appear primarily desirable to establish a suitable counting method for calculating the Miner sum $M_{Fd}$ for non-constant-amplitude stresses. Various counting methods are known, more particularly used in connection with aircraft design. These are, however, applicable to materials used in the manufacture of aircraft. It will therefore be necessary, starting from the available knowledge of concrete fatigue as well as such knowledge yet to be acquired, to undertake an evaluation and possible modification of existing methods of counting.

Furthermore, it is desirable to undertake research into the Miner number $M_{fd}$ for stress-time relations which can be described only with the aid of statistical methods, in order thus to obtain a further link-up with random loads such as those which occur in civil engineering. In connection with this it is also necessary to establish the degree of damage in relation to the type of loading in the case where Miner's rule does not give an accurate description of the damage process. Since the constant-amplitude tests have shown (see chapter 3 and [2]) that factors such as concrete grade, hardening time and the conditions of curing and testing also affect the number of stress cycles up to failure, it appears necessary also to ascertain the effect of these factors on the Miner number.

As a start of this research, IBBC-TNO is carrying out variable-amplitude tests in which the amplitude values are described by means of asymmetric probability density functions.

IBBC-TNO has moreover developed a microprocessor based random generator, with which it is possible to produce random loadings with every desired probability density function and power spectrum.

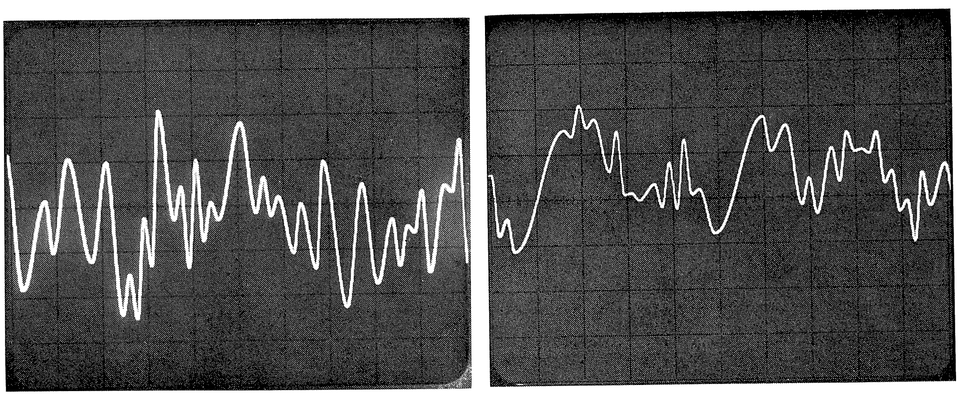

Fig. 24. Random signals with a normal distribution and a single peak spectrum (left) and a broad band spectrum (right).

The random generator consists of two seperate parts:

- A computer program for generating random numbers. These numbers are the coordinates of the peaks and valleys of a random signal with a given probability density function and a given power spectrum.
  The numbers are stored on a magnetic tape cassette with a storage capacity of some 100000 peaks and valleys.
- A micro-processor system for reconstructing the original random signal starting from the peaks and valleys on the magnetic tape.
  A micro-processor interpolates a cosine function between two succesive peaks and valleys.
  Peaks and valleys can have 256 different values, this is also true for the time difference between two successive peaks and valleys.
  The accuracy is in consequence better then 0.4%. With a 1 MHz processor a maximum frequency of 30 Hz can be generated, with a 2 MHz processor this is 60 Hz.
  In Figure 24 two scope picture of random signals are given.

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## APPENDIX I

If $N_i$ is a stochastic variable with a logarithmic-normal distribution, then it can be shown that the Miner number must also be a stochastic variable with likewise a logarithmic-normal distribution.

If Miner's rule is generally valid it must also be valid for evaluating the results $N_i$ of constant-amplitude tests. In that case the Miner number is:

$$M = \frac{N_i}{m(N_i)} \tag{I.1}$$

In formula (I.1) $m(N_i)$ is a constant and $N_i$ is logarithmic-normal distributed. The value $M$ is in that case also logarithmic-normal distributed.

## APPENDIX II

Basing oneself on the results of constant-amplitude tests formulas can be given for the mean value and the standard deviation of the Miner number.

From appendix I it is known that the Miner number $M$ has a logarithmic-normal distribution. So it is possible to give formulas for $m(M)$, $m(\log M)$, $s(M)$ and $s(\log M)$. As in calculating the Miner sum a value will be found that is related to $m(M)$ and directly related to a more familiar dispersion $s(f'_{bu})$, only formulas will be given for $m(M)$ and $s(\log M)$.

From constant-amplitude tests it is known that $N_i$ has a logarithmic-normal distribution. So $\log N_i$ has a normal distribution.

The median value for $N_i$ can be found from:

$$m(N_i) = \int_{-\infty}^{\infty} N_i \frac{1}{\sqrt{2\pi} \cdot s(\log N_i)} \cdot \exp\left[-\tfrac{1}{2}\left[\frac{\log N_i - m(\log N_i}{s(\log N_i)}\right]^2\right] =$$

$$= \int_{-\infty}^{\infty} \exp[\ln 10 \cdot \log N_i] \frac{1}{\sqrt{2\pi} \cdot s(\log N_i)} \cdot \exp\left[-\tfrac{1}{2}\left[\frac{\log N_i - m(\log N_i)}{s(\log N_i)}\right]^2\right]$$

After rearranging it follows:

$$m(N_i) = \exp[m(\log N_i) \cdot \ln 10 + \tfrac{1}{2} \ln^2 10 \cdot s^2 (\log N_i)] \times$$

$$\times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \cdot s(\log N_i)} \cdot \exp\left[\frac{\{\log N_i - m(\log N_i) - \ln^2 10 \cdot s^2(\log N_i)\}^2}{2s^2(\log N_i)}\right] =$$

$$= 10^{[m(\log N_i) + 1.15 s^2 (\log N_i)]} \tag{II.1}$$

As the Miner sum, calculated from a constant-amplitude test, has the value:

$$M = \frac{n}{10^{m(\log N_i)}} = n \cdot 10^{-m(\log N_i)} \tag{II.2}$$

the median value of $M$ will be

$$m(M) = 10^{[m(\log N_i) + 1.15 s^2(\log N_i) - m(\log N_i)]} = 10^{1.15 s^2(\log N_i)} \tag{II.3}$$

The standard deviation $s(\log M)$ follows directly from the Miner sum:

$$M = \frac{n}{10^{m(\log N_i)}} \tag{II.4}$$

In II.4 is $10^{m(\log N_i)}$ constant, $n$ is a stochastic variable. In the case of a constant-amplitude test $n = N_i$:

$$M = \frac{N_i}{10^{m(\log N_i)}} \log M = \log N_i - \log m(\log N_i) \tag{II.5}$$

As a $\log N_i$ has a normal distribution with a standard deviation $s(\log N_i)$, the standard deviation of $\log M$ has to be:

$$s(\log M) = s(\log N_i) \tag{II.6}$$