HERON contains contributions based mainly on research work performed in I.B.B.C. and STEVIN and related to strength of materials and structures and materials science.

Jointly edited by:

STEVIN-LABORATORY
of the Department of
Civil Engineering of the
Delft University of Technology,
Delft, The Netherlands
and
I.B.B.C. INSTITUTE TNO
for Building Materials
and Building Structures,
Rijswijk (ZH), The Netherlands.

HERON

vol. 24
1979
no. 4

Contents

A concrete beam reinforced with bars and steel fibres in pure bending

Prof. Dr.-Ing. H. W. Reinhardt

1 Introduction

In a bar- and steel-fibre-reinforced concrete beam all materials involved contribute to the load bearing capacity of the cross-section. Bar and fibre reinforcement provide the tensile strength, whereas the concrete is responsible for the compressive force. The stress distribution depends on the mechanical properties of the materials. In regard to the strain distribution the well-known assumption is made that the strain distribution is linear [Navier].

This assumption is correct in the uncracked stage but becomes an approximation in the cracked stage. For the sake of simplicity it will be used for both stages.

Using this assumption it is a question of the elastic moduli which determines the stress distribution in the uncracked stage. After the tensile zone of the beam has been cracked, the bond properties of the reinforcing bars, the pull-out behaviour of the fibres and the stress-strain diagram of steel and concrete govern the deflection behaviour and the bearing capacity as well.

In the following sections, three cases will be discussed: the linear elastic cracked stage, the non-linear elastic cracked stage and the non-linear elastic plastic cracked stage. In all cases it is assumed that the compression zone of the beam does not fail. The purpose of this chapter is to establish rather simple formulas for the analysis of beam test results or to show how fibres contribute to the load bearing capacity of a bar-reinforced beam.

2 The linear elastic cracked stage

In this stage concrete in compression and bar reinforcement in tension are supposed to exhibit linear elastic behaviour with the elastic moduli E_c and E_s, respectively. The stress-strain diagram of the fibre-reinforced concrete in tension is modelled by a rigid plastic behaviour according to Fig. 1. The "yield stress" σ_y is at this moment not yet known, but it will be determined in experimental investigation. It is thought to be a

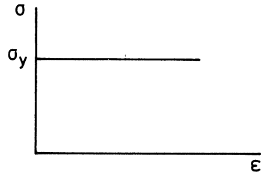

Fig. 1. Rigid-plastic behaviour as assumed for fibre reinforced concrete in tension.

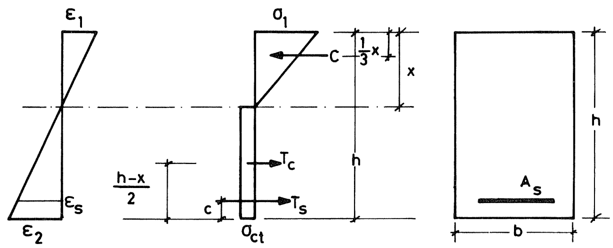

Fig. 2. Strain and stress distribution in a cross-section.

fraction of the tensile strength of the fibre concrete. This material schematization is an approximation of the real cracking behaviour which depends strongly on the amount of fibres, the type, the geometry, and the distribution of the fibres [Hannant 1978]. But for concrete mixes which are workable and useful for technical purposes, this rigid plastic approximation is justified. Experimental results [Shah et al., 1978] confirm this statement.

The tension stiffening effect of the concrete in the tensile zone will not be considered separately. It is assumed to be already covered by the rigid plastic behaviour of the fibre concrete. Using the materials properties as mentioned above, the strain and stress distribution in a rectangular cross-section becomes as follows, Fig. 2.

The upper strain ε_1 leads to a compressive stress σ_1 whereas the lower strain leads to a tensile stress σ_{ct} in the concrete. Because of the relation of Fig. 1, the tensile stress block is constant over the height of the tensile zone. The stress in the bar-reinforcement follows from the strain and the geometry.

The acting forces in the cross-section are as follows:

- compressive force $C = \frac{1}{2} b x E_c \varepsilon_1$ (1)
- tensile forces $T_c = b(h - x)\sigma_{ct}$ (2)

$$T_s = A_s E_s \varepsilon_s \tag{3}$$

with constant elastic moduli E_c and E_s.

The steel strain ε_s and the height x of the compressive zone can be derived from Fig. 2 and become

$$E_s = \left(1 - \frac{c}{h}\right)(\varepsilon_2 - \varepsilon_1) + \varepsilon_1 \tag{4}$$

and

$$x = \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2} h \tag{5}$$

where ε_1 is a negative and ε_2 is a positive strain.

The equilibrium of forces demands

$$C + T_c + T_s = 0 \tag{6}$$

Using eqs. (1) through (5) and inserting reinforcement ratio $\rho = A_s/b \cdot h$, eq. (6) becomes

$$\frac{1}{2}\frac{\varepsilon_1^2}{\varepsilon_1 - \varepsilon_2} E_c + \left[1 - \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2}\right] \sigma_{ct} + \rho E_s \left[\left(1 - \frac{c}{h}\right)(\varepsilon_2 - \varepsilon_1) + \varepsilon_1\right] = 0 \quad (7)$$

Assumed, that in a test program the quantities E_c, E_s, ρ, c, h are given and the top and bottom strains ε_1 and ε_2 have been measured, then the tensile stress in the fibre concrete can be calculated.

Another relation which must also hold true is the equilibrium of the external and internal moment:

$$M_{\text{ex}} = M_{\text{int}} \quad (8)$$

With the geometry of Fig. 2 M_{int} becomes

$$M_{\text{int}} = T_c \left(h - \frac{h - x}{2} - \frac{x}{3}\right) + T_s \left(h - c - \frac{x}{3}\right) \quad (9)$$

Usings eqs. (2) through (5) the moment equilibrium gets the form

$$\frac{M_{\text{ex}}}{bh^2} = \sigma_{ct} \frac{\varepsilon_2}{\varepsilon_2 - \varepsilon_1}\left[\frac{1}{2} + \frac{1}{6}\frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2}\right] + \rho E_s \left[\left(1 - \frac{c}{h}\right)(\varepsilon_2 - \varepsilon_1) + \varepsilon_1\right] \cdot \left[\left(1 - \frac{c}{h}\right) - \frac{1}{3}\frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2}\right] \quad (10)$$

This relation allows calculation of the tensile stress σ_{ct} if M_{ex}, E_s, b, c, h, and ρ are given and the strains ε_1 and ε_2 have been measured. Contrary to eq. (7) the knowledge of the elastic modulus of concrete is not required. In fact, the eqs. (7) and (10) are independent of each other, which makes it possible to calculate another quantity or to drop one of the measurements.

This is for instance the case if the surface strains are not measured, but instead of these only the mid span deflection is determined in the experiment. For the mid span deflection in a pure bending zone the well-known expressions are valid with the notation of Fig. 3.

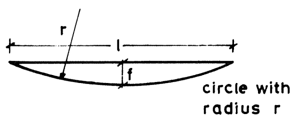

Fig. 3. Deflection of a pure bending zone.

The curvature is

$$\frac{l}{r} = \kappa = \frac{8f}{l^2 + 4f^2} \quad (11)$$

which becomes for $4f^2 \ll l^2$

$$\kappa = \frac{8f}{l^2} \quad (12)$$

On the other hand, the curvature is in terms of strains

$$\kappa = \frac{\varepsilon_2 - \varepsilon_1}{h} \tag{13}$$

Inserting eq. (13) into (7) the equilibrium of forces becomes

$$\frac{1}{2}\frac{\varepsilon_1^2}{\kappa h}E_c - \left[1 + \frac{\varepsilon_1}{\kappa h}\right]\sigma_{ct} - \rho E_s[(h-c)\kappa + \varepsilon_1] = 0 \tag{14}$$

Similarly eq. (10) changes to

$$\frac{M_{ex}}{bh^2} = \sigma_{ct}\frac{\kappa h + \varepsilon_1}{\kappa h}\left[\frac{1}{2} - \frac{1}{6}\frac{\varepsilon_1}{\kappa h}\right] + \rho E_s[(h-c)\kappa + \varepsilon_1]\left[\left(1 - \frac{c}{h}\right) + \frac{1}{3}\frac{\varepsilon_1}{\kappa h}\right] \tag{15}$$

With knowledge of M_{ex}, E_c, E_s, b, h, ρ, and κ it is possible to solve this set of equations in order to get the tensile stress σ_{ct} and the compression strain ε_1. The way of solution is enclosed in the Appendix.

3 The non-linear elastic cracked stage

Up till now the concrete (in compression) and steel were assumed to show a linear elastic behaviour. In fact, this is true up to approximately a third of the concrete strength. Beyond this limit the σ-ε-line is curved. Fig. 4 shows the strain and stress distribution with the difference from Fig. 2 that the distance e_1 of the compressive force is no longer $\frac{1}{3}x$ but depends on the shape of the σ-ε-diagram of concrete.

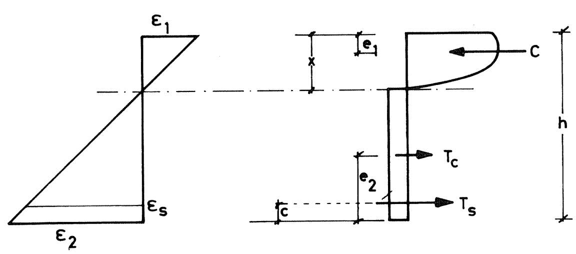

Fig. 4. Non-linear elastic cracked stage.

Eq. (7) can be written as

$$\frac{\varepsilon_1^2}{\varepsilon_1 - \varepsilon_2} f(\varepsilon_1) + \left[1 - \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2}\right]\sigma_{ct} + \rho E_s\left[\frac{h-c}{h}(\varepsilon_2 - \varepsilon_1) + \varepsilon_1\right] = 0 \tag{16}$$

$f(\varepsilon_1)$ can be taken from graphs or formulas [Leonhardt, 1973]. For the linear case it is $\frac{1}{2}E_c$ and for a parabolic σ-ε-line it becomes $\frac{2}{3}E_{c(\varepsilon=\varepsilon_1)}$.

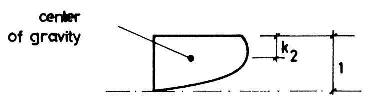

Fig. 5. Non-linear compressive stress distribution.

Eq. (10) must also be adjusted to the new center of gravity. With the coefficients k_1 and k_2, according to Fig. 5, eq. (15) becomes

$$\frac{M_{ex}}{bh^2} = \sigma_{ct} \frac{\varepsilon_2}{\varepsilon_2 - \varepsilon_1} \left[\frac{1}{2} + k_1 \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2}\right] + \rho E_s \left[\left(1 - \frac{c}{h}\right)(\varepsilon_2 - \varepsilon_1) + \varepsilon_1\right] \cdot \left[\left(1 - \frac{c}{h}\right) - k_2 \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2}\right] \quad (17)$$

In this relation $k_1 = \frac{1}{2} - k_2$. For the linear case, $k_2 = \frac{1}{3}$; for the parabolic shape of the σ-ε-line $k_2 = \frac{3}{8}$. If κ is known instead of ε_1 and ε_2 the same procedure can be applied as in part 2.

4 The non-linear elastic plastic cracked stage

If the stress in the reinforcing steel reaches the yield point the elastic stage changes into the plastic stage. All equations derived until now can be maintained if the modulus of elasticity of steel E_s is correctly interpreted. Fig. 6 illustrates that E_s is a constant value up to the yield point and starts to be a variable beyond this point. That means that E_s depends on the state of strain and the type of steel used.

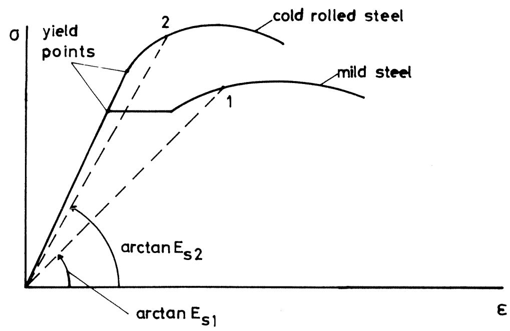

Fig. 6. Definition of E_s in the post yield stage.

5 Experimental verification

In the test program which will be described further in this issue of HERON, static bending tests have been carried out on fibre- and bar-reinforced concrete beams using three types of fibres with three different amounts of fibres. Also three different amounts of bar reinforcement were used representing a minimum, a medium and a high reinforcement ratio. The cylinder strength of the concrete was about 40 N/mm^2 and the elastic modulus of the concrete has a value of about $26 \cdot 10^3$ N/mm^2.

From the test results the following fictitious tensile stresses in the concrete have been calculated (pl/d = volume aspect ratio with $p$vol.% of fibres, l length of the fibres, d diameter of the fibres):

Table 1. Tensile stress in fibre concrete as calculated.

Type of fibre	ρ	σ_{ct} in N/mm²	
pl/d%	%	uncracked	cracked
straight	0.17	1.12	2.20
76	0.75	1.15	2.54
	2.09	1.22	2.86
hooked	0.17	0.87	2.99
69	0.75	-	1.84
	2.09	0.85	1.41
paddled	0.17	0.67	2.90
96	0.75	-	2.25
	2.09	0.76	1.47

From the results [Reinhardt, 1978] it could clearly be distinguished between the uncracked and the cracked stage. In the uncracked stage the calculated tensile concrete stress is much less than in the cracked stage. This can be interpreted easily by the fact that the outmost fibre only reaches a strain of 0.25‰, which means that the assumed stress distribution is not quite correct. A triangular distribution in the tensile zone would be a better approach.

When cracks are formed and the tensile strain reaches values up to 2‰, the assumed stress distribution definitely corresponds much closer with the real behaviour. In this case, the calculated tensile stress is equal to 100% of the uniaxial tensile strength for the straight fibres and 85% of the uniaxial tensile strength of the concrete with hooked and paddled fibres.

6 Conclusions

A simple model has been presented by which account can be taken of the fibre contribution in a bar-reinforced beam subjected to pure bending. Comparing experimental and theoretical results it turned out that for straight fibres 100% of the uniaxial tensile strength and for hooked and paddled fibres 85% of the uniaxial strength can be used to advantage in the calculation of the load bearing capacity of a beam.

7 References

HANNANT, D. J. (1978). Fibre cements and fibre concretes. John Wiley and Sons, Chichester, New York.

LEONHARDT, F. (1973). Vorlesungen über Massivbau. Vol. 1. Springer Berlin, New York.

REINHARDT, H. W. (1978). Contribution of the fibres to the load bearing capacity of a bar- and fibre-reinforced concrete beam. Stevin Report 5-78-9, Delft, Holland.

SHAH, S. P., P. STROEVEN, D. DALHUISEN and P. VAN STEKELENBURG (1978). Complete stress-strain curves for steel fibre reinforced concrete in uniaxial tension and compression. RILEM Symposium Sheffield.

8 Appendix

Determination of ε_1 and σ_{ct} from κ only

The two relevant equations are

$$\frac{\varepsilon_1^2}{\kappa h} f(\varepsilon_1) - \left[1 + \frac{\varepsilon_1}{\kappa h}\right] \sigma_{ct} - \rho E_s[(h-c)\kappa + \varepsilon_1] = 0 \quad \text{(A1)}$$

$$\sigma_{ct} \frac{\kappa h + \varepsilon_1}{\kappa h} \left[\frac{1}{2} - k_1 \frac{\varepsilon_1}{\kappa h}\right] + \rho E_s[(h-c)\kappa + \varepsilon_1] \left[\left(1 - \frac{c}{h}\right) + k_2 \frac{\varepsilon_1}{\kappa h}\right] - \frac{M_{\text{ex}}}{bh^2} = 0 \quad \text{(A2)}$$

Isolate σ_{ct} in eq. (A1)

$$\sigma_{ct} = \frac{\kappa h}{\kappa h + \varepsilon_1} \left\{\frac{\varepsilon_1^2}{\kappa h} f(\varepsilon_1) - \rho E_s[(h-c)\kappa + \varepsilon_1]\right\} \quad \text{(A3)}$$

Replace σ_{ct} in eq. (A2) by right hand side of eq. (A3)

$$\left[\frac{1}{2} - k_1 \frac{\varepsilon_1}{\kappa h}\right] \left\{\frac{\varepsilon_1^2}{\kappa h} f(\varepsilon_1) - \rho E_s[(h-c)\kappa + \varepsilon_1]\right\} +$$

$$+ \rho E_s[(h-c)\kappa + \varepsilon_1] \left[\left(\frac{h-c}{h}\right) + k_2 \frac{\varepsilon_1}{\kappa h}\right] - \frac{M_{\text{ex}}}{bh^2} = 0 \quad \text{(A4)}$$

Rearrange eq. (A4) making use of $k_1 = \frac{1}{2} - k_2$

$$-\frac{k_1 f(\varepsilon_1)}{\kappa^2 h^2} \varepsilon_1^3 + \left[\frac{1}{2} \frac{f(\varepsilon_1)}{\kappa h} + \frac{1}{2} \frac{\rho E_s}{\kappa h}\right] \varepsilon_1^2 + \left[-\tfrac{1}{2}\rho E_s + \tfrac{3}{2}\rho E_s \frac{h-c}{h}\right] \varepsilon_1 +$$

$$+ \left[-\frac{\rho E_s}{2} (h-c)\kappa + \rho E_s \frac{(h-c)^2}{h} \kappa - \frac{M_{\text{ex}}}{bh^2}\right] = 0 \quad \text{(A5)}$$

Simplify the equation and get

$$-\frac{k_1 f(\varepsilon_1)}{\kappa^2 h^2} \varepsilon_1^3 + \frac{1}{2\kappa h} [f(\varepsilon_1) + \rho E_s] \varepsilon_1^2 + \tfrac{1}{2}\rho E_s \left[-1 + 3 \frac{h-c}{h}\right] \varepsilon_1 +$$

$$+ \rho E_s \kappa \left[-\frac{h-c}{2} + \frac{(h-c)^2}{h}\right] - \frac{M_{\text{ex}}}{bh^2} = 0 \quad \text{(A6)}$$

From this cubic equation ε_1 can be calculated for a measured κ. All other quantities like $f(\varepsilon_1)$, ρ, E_s, h, c, M_{ex} must be given.

Generally there are three roots of eq. (A6), one of which is the right one. That ε_1 must be put in eq. (A3) in order to get the contribution of the fibres σ_{ct}.