HERON contains contributions based mainly on research work performed in I.B.B.C. and STEVIN and related to strength of materials and structures and materials science.

Jointly edited by:

STEVIN-LABORATORY
of the Department of Civil Engineering of the Delft University of Technology, Delft, The Netherlands
and
I.B.B.C. INSTITUTE TNO
for Building Materials and Building Structures, Rijswijk (ZH), The Netherlands.

EDITORIAL BOARD:

J. Witteveen, *editor in chief*
G. J. van Alphen
M. Dragosavić
H. W. Reinhardt
A. C. W. M. Vrouwenvelder
L. van Zetten

Secretary:

G. J. van Alphen
Stevinweg 1
P.O. Box 5048
2600 GA Delft, The Netherlands

HERON vol. 25 1980 no. 1

Contents

GENERAL DERIVATION OF HARDENING FORMULAS

Ir. A. K. de Groot
Ir. G. M. A. Kusters

I.B.B.C. Institute TNO

Notation

F	yield function ($F = F_1 + F_2$)
F_2	part of F which depends solely on σ_y
$F_1(f)$	rest of F
Q	a function to which the plastic strain increments are perpendicular $Q = F$ (associative flow) $Q \neq F$ (non-associative flow)
n	degree of the function F
H	hardening modulus
A	effect of hardening on elasto-plastic strain relation
C_y	material constant of the yield function
f_t	uniaxial tensile strength
W_p	dissipated energy
$[\,], [\,]^{-1}, [\,]^T$	matrix, the inverse and the transpose
$\{\,\}, \{\,\}^T$	column vector and row vector respectively
$[S_e]$	linear elastic stress-strain relation
$[S_{ep}]$	elasto-plastic stress-strain relation
ε_{ij}	total strain tensor
ε_{ij}^e	elastic strain tensor
ε_{ij}^p	plastic strain tensor
ε_y	uniaxial total strain ($\varepsilon_y = \varepsilon_y^e + \varepsilon_y^p$)
ε_y^p	uniaxial equivalent plastic strain
ε_y^e	uniaxial elastic strain
ε_m^p	plastic volume strain
σ_y	uniaxial yield stress
$\sigma_{y①}$	initial uniaxial yield stress
σ_u	ultimate uniaxial yield stress
$\sigma_1, \sigma_2, \sigma_3$	principal stresses
σ_4	auxiliary quantity (see Appendix B)
C	constant in Table B3
σ_{okt}	first stress invariant $\sigma_{okt} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$
$\bar{\sigma}$	second stress invariant $\bar{\sigma} = \sqrt{\frac{1}{6}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}} = \sqrt{\frac{1}{2} s_{ij} s_{ij}}$
Φ	third stress invariant $\Phi = \frac{1}{3}\sin^{-1}\left[-\frac{3\sqrt{3}}{2}\frac{J_3}{\bar{\sigma}^3}\right]$ with $-\frac{1}{6}\pi \leq \Phi \leq \frac{1}{6}\pi$ where: $J_3 = (\sigma_1 - \sigma_{okt})(\sigma_2 - \sigma_{okt})(\sigma_3 - \sigma_{okt})$
d	scalar
α	interpolation factor (see Appendix C)
k	interpolation factor (see Appendix B)

φ	angle in hydrostatic section (see Appendix C)
δ_{ij}	Kronecker δ; $i=j \rightarrow \delta=1$; $i \neq j \rightarrow \delta=0$
β_1, β_2	auxiliary quantities in Table 1
θ	friction (see equation A1)
c	cohesion (see equation A2)

GENERAL DERIVATION OF HARDENING FORMULAS

Summary

The usefulness of the elasto-plastic arithmetical model for describing the behaviour of a material such as concrete depends to a great extent on the hardening rules to be applied. Without going into the background of these rules and without expressing a preference, it will be endeavoured in this article to give a general derivation of the hardening formulas, as opposed to the usual procedure of treating the hardening rules for each particular yield criterion. After the general derivation, the formulas are further developed for a number of well-known yield criteria in Appendix A. Furthermore, in Appendices B and C, as in Chapter 4, the possibility of interpolation between different hardening diagrams is discussed, without implying that the interpolation methods presented here are necessarily the most suitable. The need for some form of interpolation must be recognized, however. In many publications this problem is ignored, thus creating the impression that these are not matters likely to cause any difficulties. Finally, in Appendix D it is pointed out that the manner in which a particular yield criterion is formulated may sometimes make arithmetical treatment more difficult.

General derivation of hardening formulas

1 Introduction

The plastic behaviour of a material such as steel can be described with the elasto-plastic material model which is characterized by:

1. An initial yield criterion, defining the elastic limit of the material.
2. A flow rule, relating the plastic strain increments to the stresses and stress increments.
3. A hardening rule, used to establish conditions for subsequent yield from a plastic state.

For a material such as concrete the above-mentioned characterization is likewise a good approximation of the actual behaviour, except that, in addition, it is necessary to pay special attention to cracking and brittle fracture, which can be done by the introduction of:

4. A tension cut-off criterion.
5. A crush criterion.

The present article is more particularly concerned with point 3, the hardening rule, while point 1 will also be considered.

With regard to point 2, the flow rule, it is noted only that Drucker's postulate will be employed. Points 4 and 5, though of major importance to a satisfactory description of the behaviour of concrete, will not be considered in this article (for point 4 see, for example, [8]).

With reference to hardening, a distinction is usually drawn between *isotropic* hardening (Fig. 1a) and *kinematic* hardening (Fig. 1b).

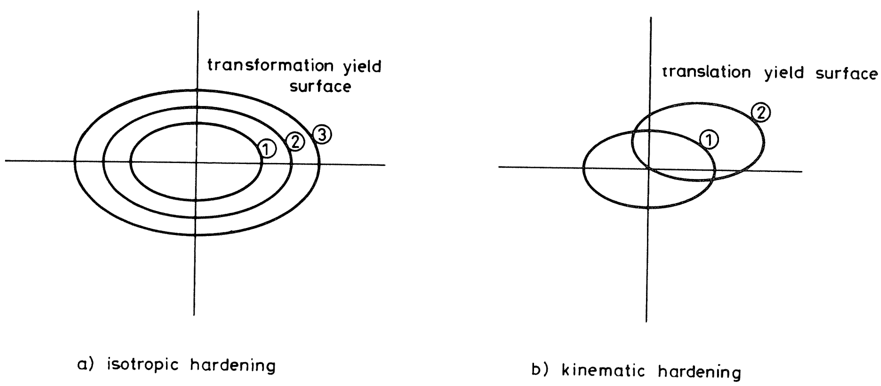


Fig. 1. The difference between isotropic and kinematic hardening.

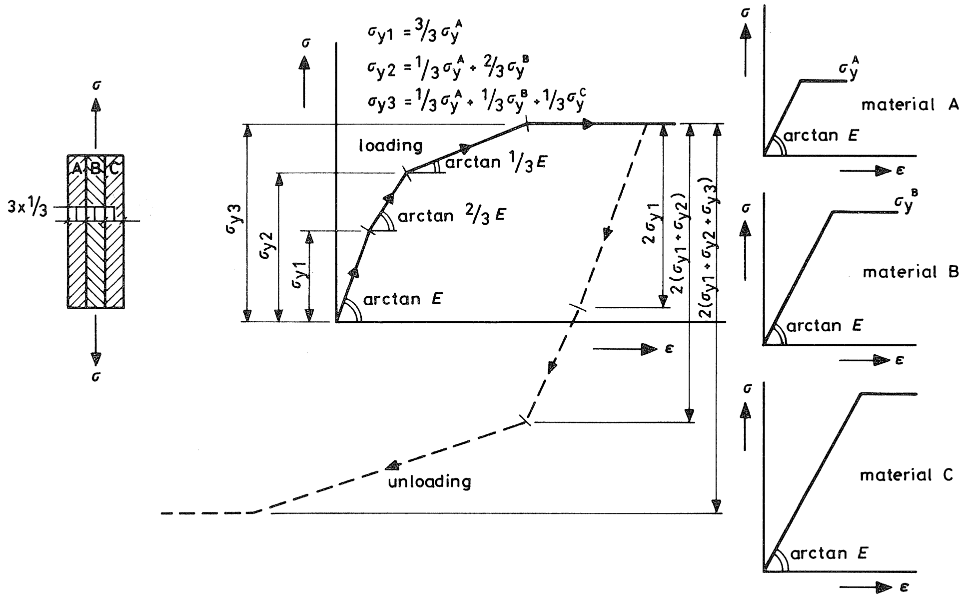


Fig. 2. An example of the fraction model for cyclic loading.

Both methods of description involve some drawbacks, but these can be overcome by making use of the fraction model [1], known also as the overlay model (for example, see [5]). In this model a material particle is conceived as being composed of a number of "fractions" connected in parallel, to each of which can be assigned properties differing from those of the others. This conception can be visualized with the aid of Fig. 2, relating to a uniaxially loaded bar. The bar is conceived, for example, as composed of three materials (A, B and C) with ideal elasto-plastic behaviour. It is not difficult to see that, by virtue of appropriate choice of yield points and moduli of elasticity, the model can, in a simple manner, be adapted as well as possible to reality.

So, in general we can say that by appropriate choice of the number of fractions and the properties applicable per fraction, such as the yield condition, the initial yield stress and the isotropic hardening rule, any material behaviour can be simulated as closely as possible.

Of course, by choosing just one fraction, we arrive at isotropic hardening for the whole bar. However, if necessary, with the aid of the fraction model it is possible also to describe kinematic hardening by making a suitable choice of the number of fractions, their size and the isotropic hardening rules applied.

This means that, if the fraction model is employed, the kinematic hardening models of, for example, Prager [2] and Ziegler [3], [4] are no longer needed and that attention may be confined to the isotropic hardening model. When using the last-mentioned model, there are two customary ways in which the degree of hardening is rated:

- strain hardening
- work hardening

In this article, strain hardening denotes that the degree of hardening depends on the second invariant of the plastic strains. By work hardening is understood that the degree of hardening depends on the dissipated energy. This should be emphatically pointed out, because in the literature these two approaches are often lumped together as "work hardening".

For both methods a general formulation of the hardening rules will now be given wich is independent of the yield criterion under consideration (provided that this criterion fulfils certain conditions). Next, for a number of known criteria, the general formulation is further elaborated to some extent. For convenience of presentation some derivations are included in appendices to the article. First, however, a chapter will be devoted to general aspects of the elasto-plastic strain relation.

This article has more particularly been prompted by a study undertaken within the context of activities of CUR Committee A 26 "Concrete mechanics" [9], with the object of presenting an overall review of the various constitutive material models as encountered in the literature.

2 The elasto-plastic stress-strain relation

In general, yield criteria can be written in the following form:

$$\boxed{F = f(\{\sigma\}, \sigma_y) - C_y \sigma_y^n = 0} \tag{1}$$

With this notation a general derivation of the hardening rule can be given. On introduction of:

$$F_1 = f(\{\sigma\}, \sigma_y) \tag{2}$$

and:

$$F_2 = C_y \sigma_y^n \tag{3}$$

the yield criterion can alternatively be written as follows:

$$\boxed{F = F_1 - F_2 = 0} \tag{4}$$

The function F_2 is dependent only on the yield stress σ_y, which occurs to the degree n, and a constant C_y. The function F_1 is dependent, except possibly also in the yield stress σ_y, on the stresses $\{\sigma\}$. In the case of work hardening this function will (as shown in Section 3.3) have to be homogeneous and of the same degree n as that of σ_y in F_2. With reference to σ_y it is furthermore to be noted that it varies between the initial yield stress $\sigma_y = o_{y0}$ and the maximum yield stress $\sigma_y = \sigma_u$.

The required relation between stresses and strains is:

$$\{d\sigma\} = [S_{ep}]\{d\varepsilon\} \tag{5}$$

where:

$$\{d\sigma\} = \begin{Bmatrix} d\sigma_x \\ d\sigma_y \\ d\sigma_z \\ d\tau_{xy} \\ d\tau_{yz} \\ d\tau_{zx} \end{Bmatrix} \quad \text{and} \quad \{d\varepsilon\} = \begin{Bmatrix} d\varepsilon_x \\ d\varepsilon_y \\ d\varepsilon_z \\ d\gamma_{xy} \\ d\gamma_{yz} \\ d\gamma_{zx} \end{Bmatrix}$$

According to the above-mentioned relation, the total strain vector $\{\varepsilon\}$ increases by the increment vector $\{d\varepsilon\}$ which is composed of an elastic and plastic portion, as follows:

$$\{d\varepsilon\} = \{d\varepsilon^e\} + \{d\varepsilon^p\} \tag{6}$$

So long as the material is still elastic, the second term is zero, i.e. $\{d\varepsilon^p\} = 0$, so that $\{d\varepsilon\} = \{d\varepsilon^e\}$ and therefore equation (5) can be written as:

$$\{d\sigma\} = [S_e]\{d\varepsilon\} \tag{7}$$

where $[S_e]$ denotes the elastic stress-strain relation.

It will now be shown how the elasto-plastic strain relation $[S_{ep}]$ can be obtained. First, however, it is necessary to discuss the distinction between associative and non-associative flow. The term associative flow is employed when it is assumed that Drucker's postulate is valid, which states that the plastic strain increment is perpendicular to the yield surface F, so that:

$$d\varepsilon^p = d\lambda \left\{\frac{\partial F}{\partial \sigma}\right\} \tag{8}$$

or:

$$\begin{Bmatrix} d\varepsilon_x^p \\ d\varepsilon_y^p \\ d\varepsilon_z^p \\ d\gamma_{xy}^p \\ d\gamma_{yz}^p \\ d\gamma_{zx}^p \end{Bmatrix} = d\lambda \begin{Bmatrix} \dfrac{\partial F}{\partial \sigma_x} \\ \dfrac{\partial F}{\partial \sigma_y} \\ \dfrac{\partial F}{\partial \sigma_z} \\ \dfrac{\partial F}{\partial \tau_{xy}} \\ \dfrac{\partial F}{\partial \tau_{yz}} \\ \dfrac{\partial F}{\partial \tau_{zx}} \end{Bmatrix}$$

Should Drucker's postulate not be valid, then a function Q can be established to which the normality principle is applicâble, so that in more general terms the following can be stated:

$$\{d\varepsilon^p\} = d\lambda \left\{\frac{\partial Q}{\partial \sigma}\right\} \tag{9}$$

In the case represented by equation (9) there is non-associative flow.

The stress-strain relation will now be elaborated for non-associative flow ($Q \neq F$). If it is desired to base oneself on associative flow, it is merely necessary subsequently to put $Q = F$.

Equation (9) together with equation (7) gives:

$$\{\mathrm{d}\varepsilon\} = \{\mathrm{d}\varepsilon^e\} + \mathrm{d}\lambda \left\{\frac{\partial Q}{\partial \sigma}\right\} \tag{10}$$

for which, with equation (7), we can write:

$$\{\mathrm{d}\varepsilon\} = [S_e]^{-1}\{\mathrm{d}\sigma\} + \mathrm{d}\lambda \left\{\frac{\partial Q}{\partial \sigma}\right\} \tag{11}$$

On premultiplication of the left and right sides of this equation by

$$\left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]$$

we obtain:

$$\left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]\{\mathrm{d}\varepsilon\} = \left\{\frac{\partial F}{\partial \sigma}\right\}^T \{\mathrm{d}\sigma\} + \mathrm{d}\lambda \left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e] \left\{\frac{\partial Q}{\partial \sigma}\right\} \tag{12}$$

If the material yields, then $F = 0$; if moreover no unloading takes place, then $\mathrm{d}F = 0$, so that we obtain from equation (4):

$$\mathrm{d}F = \left\{\frac{\partial F}{\partial \sigma}\right\}^T \{\mathrm{d}\sigma\} + \frac{\partial F}{\partial \sigma_y}\mathrm{d}\sigma_y = 0 \tag{13}$$

Where $(\partial F/\partial \sigma)\mathrm{d}\sigma_y$ represents the effect of hardening.

Before further working out these expressions, we shall first, following the example of Nayak and Zienkiewicz [5], [6] introduce the auxiliary quantity A, as follows:

$$A = -\frac{1}{\mathrm{d}\lambda}\frac{\partial F}{\partial \sigma_y}\mathrm{d}\sigma_y \tag{14}$$

This quantity A comprises all the influences that determine the degree of hardening. Thus, for the case of an ideal plastic material (i.e., no hardening): $A = 0$. With equation (14) we can now write equation (13) as follows:

$$\mathrm{d}F = \left\{\frac{\partial F}{\partial \sigma}\right\}^T \mathrm{d}\sigma - A\,\mathrm{d}\lambda = 0 \tag{15}$$

On substituting this into equation (12) and rearranging, we obtain:

$$\boxed{\mathrm{d}\lambda = \frac{\left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]\{\mathrm{d}\varepsilon\}}{A + \left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]\left\{\frac{\partial Q}{\partial \sigma}\right\}}} \tag{17}$$

Substitution of this into equation (11) gives:

$$\{d\varepsilon\} = [S_e]^{-1}\{d\sigma\} + \frac{\left\{\frac{\partial Q}{\partial \sigma}\right\}\left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]}{A + \left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]\left\{\frac{\partial Q}{\partial \sigma}\right\}} \cdot \{d\varepsilon\} \tag{18}$$

Premultiplication of the left and right sides by $[S_e]$ followed by some rearranging now gives:

$$\boxed{\{d\sigma\} = \left[[S_e] - \frac{[S_e]\left\{\frac{\partial F}{\partial \sigma}\right\}\left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]}{A + \left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]\left\{\frac{\partial Q}{\partial \sigma}\right\}}\right]\{d\varepsilon\}} \tag{19}$$

so that we thus have the required relation:

$$\{d\sigma\} = [S_{ep}]\{d\varepsilon\} \tag{5}$$

where:

$$[S_{ep}] = [S_e] - \frac{[S_e]\left\{\frac{\partial F}{\partial \sigma}\right\}\left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]}{A + \left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]\left\{\frac{\partial Q}{\partial \sigma}\right\}} \tag{20}$$

In this equations, as already stated, A is an auxiliary quantity with which the effect of hardening is taken into account. This will be further considered in the next chapters.

A variant formulation of equation (20) is as follows:

$$[S_{ep}] = [S_e] - (1-h)\,\frac{[S_e]\left\{\frac{\partial F}{\partial \sigma}\right\}\left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]}{\left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]\left\{\frac{\partial Q}{\partial \sigma}\right\}} \tag{20a}$$

where:

$$h = \frac{A}{A + \left\{\frac{\partial F}{\partial \sigma}\right\}^T [S_e]\left\{\frac{\partial Q}{\partial \sigma}\right\}} \tag{20b}$$

It should be noted that, although the elasto-plastic stress-strain relation has been derived for the case $Q \neq F$, the further consideration concerning the correct magnitude of A will be based on the assumption $Q = F$, i.e., only for associative flow.

3 Derivation of hardening formulas

3.1 *General*

According to equation (14):

$$A = -\frac{1}{\mathrm{d}\lambda}\frac{\partial F}{\partial \sigma_y}\mathrm{d}\sigma_y \tag{14}$$

where, according to equation (1):

$$\frac{\partial F}{\partial \sigma_y} = \frac{\partial F_1}{\partial \sigma_y} - nC_y\sigma_y^{n-1} \tag{21}$$

while, with reference to the hardening diagram to be dealt with in Chapter 4, we can write:

$$\mathrm{d}\sigma_y = H\mathrm{d}\varepsilon_y^p \tag{22}$$

Hence it is assumed that the degree of hardening can be described with a uniaxial σ-ε diagram in which a comparison stress σ_y is plotted against an equivalent plastic strain ε_y^p.

Substitution of equations (21) and (22) into equation (14) then gives:

$$\boxed{A = H\frac{\mathrm{d}\varepsilon_y^p}{\mathrm{d}\lambda}\left(nC_y\sigma_y^{n-1} - \frac{\partial F_1}{\partial \sigma_y}\right)} \tag{23}$$

Thus we have the formula for determining A. It is a general formula and not dependent on any particular yield criterion or hardening model (work hardening or strain hardening). This is so because the dependence on the yield criterion is incorporated in the expression $nC_y\sigma_y^{n-1} - \partial F_1/\partial\sigma_y$, while the dependence on the hardening model adopted is incorporated in the term $H\mathrm{d}\varepsilon_y^p$, where H is the tangent modulus in the hardening diagram and $\mathrm{d}\varepsilon_y^p$ is the increase of the equivalent plastic strain in this same diagram.

It will now be shown how this $\mathrm{d}\varepsilon_y^p$ is determined, first in the case of strain hardening (Section 3.2), then in the case of work hardening (Section 3.3).

3.2 *Strain hardening*

With strain hardening, the equivalent plastic strain increment is taken as proportional to the second invariant of the plastic strain increments, as follows:

$$\mathrm{d}\varepsilon_y^p = \alpha\sqrt{\tfrac{2}{3}\mathrm{d}\varepsilon_{ij}^p\,\mathrm{d}\varepsilon_{ij}^p} \tag{24}$$

where α is an interpolation coefficient which is mostly equal to 1, namely, in all cases where one hardening diagram at a time is considered. Only if, with affine hardening diagrams, interpolation is done in the manner described in Section 4.5, does this coefficient α have a value not equal to 1.

Since, according to Drucker's postulate:

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial F_1}{\partial \sigma_{ij}} \tag{25}$$

we can write equation (24) finally as follows:

$$\boxed{d\varepsilon_{y}^{p} = \alpha\, d\lambda \sqrt{\frac{2}{3} \frac{\partial F_1}{\partial \sigma_{ij}} \frac{\partial F_1}{\partial \sigma_{ij}}}} \tag{26}$$

whence, with equation (23), we obtain for A:

$$\boxed{A = \alpha H \left(n C_y \sigma_y^{n-1} - \frac{\partial F_1}{\partial \sigma_y} \right) \sqrt{\frac{2}{3} \frac{\partial F_1}{\partial \sigma_{ij}} \frac{\partial F_1}{\partial \sigma_{ij}}}} \tag{27}$$

3.3 *Work hardening*

For the increase dW_p of the total dissipated energy W_p we can write:

$$dW_p = \{d\varepsilon^p\}^T \{\sigma\} \tag{28}$$

The treatment of the subject will be based on associative flow, or equation (8), in which, according to equations (1) and (2), we can put:

$$\left\{ \frac{\partial F}{\partial \sigma} \right\} = \left\{ \frac{\partial F_1}{\partial \sigma} \right\}$$

so that equation (28) can also be written as follows:

$$dW_p = d\lambda \left\{ \frac{\partial F_1}{\partial \sigma} \right\}^T \{\sigma\} \tag{29}$$

In the further derivation, Euler's theorem will be applied. This is possible only if the function F_1 is homogeneous. Whereas this was not important in the case of strain hardening, in the following treatment of the work hardening case it is a necessary condition.

The theorem in question (see [10], for example) states that for homogeneous functions $f(\{x\})$ of n^{th} degree:

$$\left\{ \frac{\partial f}{\partial x} \right\}^T \{x\} = nf \tag{30}$$

or written differently:

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_m \frac{\partial f}{\partial x_m} = nf(x_1, x_2, \ldots x_m)$$

This may be illustrated by the following example:

$$f(x, y) = xy \ln \frac{y}{x}$$

$$\frac{\partial f}{\partial x} = y \left(-1 + \ln \frac{y}{x} \right)$$

$$\frac{\partial f}{\partial y} = x \left(1 + \ln \frac{y}{x} \right)$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = xy \left(-1 + \ln \frac{y}{x} \right) + xy \left(1 + \ln \frac{y}{x} \right) = 2xy \ln \frac{y}{x}$$

For this example: $n = 2$, and we see that indeed:

$$2xy \ln \frac{y}{x} = 2f(x, y)$$

The theorem will now be applied to the homogeneous functions F_1 in equation (2) of n^{th} degree, giving:

$$\left\{ \frac{\partial F_1}{\partial \sigma} \right\}^T \{\sigma\} + \frac{\partial F_1}{\partial \sigma_y} \sigma_y = nF_1 \tag{31}$$

Since, according to equation (4), nF_1 is equal to $nF_2 = nC_y \sigma_y^n$, equation (31) can also be written as:

$$\left\{ \frac{\partial F_1}{\partial \sigma} \right\}^T \{\sigma\} = nC_y \sigma_y^n - \frac{\partial F_1}{\partial \sigma_y} \sigma_y \tag{32}$$

Substitution of equation (32) into equation (29) gives:

$$\mathrm{d} W_p = \mathrm{d}\lambda \sigma_y \left(nC_y \sigma_y^{n-1} - \frac{\partial F_1}{\partial \sigma_y} \right) \tag{33}$$

In accordance with the definition of the hardening diagram (see Chapter 4), the following expression can be written for $\mathrm{d} W_p$:

$$\mathrm{d} W_p = \frac{1}{\alpha} \mathrm{d}\varepsilon_y^p \sigma_y \tag{34}$$

where α is again the interpolation factor discussed in Section 4.5, so that, on equating (33) and (34), we obtain:

$$\boxed{\mathrm{d}\varepsilon_y^p = \alpha \, \mathrm{d}\lambda \left(nC_y \sigma_y^{n-1} - \frac{\partial F_1}{\partial \sigma_y} \right)} \tag{35}$$

With equation (35) the strain increment $\mathrm{d}\varepsilon_y^p$ after each loading step is known. By summation of these increments we obtain the total equivalent strain ε_y^p, so that the coefficient H can be read from the hardening diagram.

Equation (23) is applicable both to strain hardening and to work hardening. The formula for the determination of $d\varepsilon_y^p$ (equation 35) is valid only in the last-mentioned case. For the case of strain hardening, see equation (26).

To conclude this section of the article, two alternative formulas for A will be given. It is emphatically pointed out that these apply only to the case of work hardening.

The first alternative formula is obtained by substituting equation (35) into equation (23), giving:

$$\boxed{A = \alpha H \left(nC_y\sigma_y^{n-1} - \frac{\partial F_1}{\partial \sigma_y}\right)^2} \tag{36}$$

The second alternative formula is obtained as the result of substituting equation (32) into equation (36):

$$A = \frac{\alpha H}{\sigma_y}\left\{\frac{\partial F_1}{\partial \sigma}\right\}^T \{\sigma\} \left(nC_y\sigma_y^{n-1} - \frac{\partial F_1}{\partial \sigma_y}\right) \tag{37}$$

This formulation is also frequently encountered in the literature, though for the special case: $\alpha = 1$, $n = 1$, $C_y = 1$ and $\partial F_1/\partial\sigma_y = 0$, so that equation (37) thus becomes:

$$A = \frac{H}{\sigma_y}\left\{\frac{\partial F_1}{\partial \sigma}\right\}^T \{\sigma\} \tag{38}$$

3.4 *Discussion of the results*

The formulas for the determination of $d\varepsilon_y^p$ and H, derived in the preceding sections, are still of a general character. For each yield criterion they can be further worked out by writing $\partial F_1/\partial\sigma_y$ and (so far as possible) $\sqrt{\frac{2}{3}\partial F/\partial\sigma_{ij}\partial F/\partial\sigma_{ij}}$ in explicit form and substituting them into the formulas.

The formulas are developed for various yield criteria in Appendix A. The results are summarized in Table 1.

It is seen there that $d\varepsilon_y^p$ and A are dependent on the yield criterion employed, which is something that must not be overlooked.

Thus, Buyukozturk [7] has erroneously based his criterion on:

$$d\lambda = d\varepsilon_y^p$$

a relation which is valid only for the Von Mises criterion formulated in the manner indicated in Table 1.

According to Table 1, Buyukozturk should have used the following formulation:

$$d\varepsilon_y^p = d\lambda\left(1 - \frac{27\sigma_{\text{okt}}}{2\sigma_y}\right) \tag{39}$$

Table 1. Summary of the hardening formulas established, embodying the auxiliary quantities β_1 and β_2.

Yield criterion	$\beta_1 =$	$\beta_2 =$
General formulation $F = \underbrace{f(\sigma_{ij}, \sigma_y)}_{F_1} - C_y \sigma_y^n = 0$	$n C_y \sigma_y^{n-1} \dfrac{\partial F_1}{\partial \sigma_y}$	$\sqrt{\frac{2}{3} \dfrac{\partial F_1}{\partial \sigma_{ij}} \dfrac{\partial F_1}{\partial \sigma_{ij}}}$
Tresca $F = 2\bar{\sigma} \cos \Phi - \sigma_y = 0$	1	$\frac{2}{3}\sqrt{3}$
Von Mises $F = \bar{\sigma}\sqrt{3} - \sigma_y = 0$	1	1
Buyukozturk [7] $F = 3\sqrt{3}\bar{\sigma}^2 + 3\sigma_y \sigma_{okt} + \frac{9}{5}\sigma_{okt}^2 - \sigma_y = 0$	$1 - \dfrac{27\sigma_{okt}}{2\sigma_y}$	$\sqrt{\frac{2}{3} \dfrac{\partial F_1}{\partial \sigma_{ij}} \dfrac{\partial F_1}{\partial \sigma_{ij}}}$ *
Buyukozturk (see variant Appendix D) $F = 27\bar{\sigma}^2 + 27\sigma_y \sigma_{okt} + \frac{81}{5}\sigma_{okt}^2 - \sigma_y^2 = 0$	$2\sigma_y - 27\sigma_{okt}$	$\sqrt{\frac{2}{3} \dfrac{\partial F_1}{\partial \sigma_{ij}} \dfrac{\partial F_1}{\partial \sigma_{ij}}}$ *
Drucker-Prager $F = \dfrac{6\sigma_{okt} \sin \theta}{(3 - \sin \theta)\sqrt{3}} + \bar{\sigma} - \dfrac{\sigma_y(1 - \sin \theta)\sqrt{3}}{3 - \sin \theta} = 0$	$\dfrac{(1 - \sin \theta)\sqrt{3}}{3 - \sin \theta}$	$\sqrt{\dfrac{8 \sin^2 \theta}{3(3 - \sin \theta)^2} + \frac{1}{3}}$
Mohr-Coulomb $F = \sigma_{okt} \sin \theta + \bar{\sigma} \cos \Phi - \bar{\sigma}/\sqrt{3} \sin \Phi \sin \theta - \frac{1}{2}\sigma_y(1 - \sin \theta)$	$\dfrac{1 - \sin \theta}{2}$	$\sqrt{\dfrac{1 + \sin^2 \theta}{3}}$

Workhardening: $d\varepsilon_y^p = \alpha \, d\lambda \beta_1$ ** , $A = \alpha H \beta_1^2$

Strainhardening: $d\varepsilon_y^p = \alpha \, d\lambda \beta_2$ ** , $A = \alpha H \beta_1 \beta_2$

* Cannot be reduced to a simple expression

** $\alpha = 1$ unless interpolation is done in accordance with section 4.5

The formula given for A in [7] is likewise affected by this. For the benefit of those readers who may wish to consult that publication, an error in differentiation committed in it should also be noted. Instead of

$$\frac{27}{2} \frac{\sigma_{okt}}{\sigma_y} \left(\frac{9J_1}{2\bar{\sigma}} \right) \text{ the expression } \frac{9\sigma_{okt}}{2\sigma_y} \left(\frac{3J_1}{2\bar{\sigma}} \right) \text{ is used there.}$$

The notation given in parentheses () is that which has been used in [7].

Finally, it should be pointed out that the formulas for the determination of $d\varepsilon_y^p$ and A will, with the exception of equation (23), change if an alternative formulation is chosen for a particular yield criterion.

As an example, consider the Von Mises criterion for work hardening. The formulation given in Table 1:

$$F = \bar{\sigma}\sqrt{3} - \sigma = 0 \tag{40}$$

(for the meaning of $\bar{\sigma}$ see the list of symbols in the appended "Notation") with $\alpha = 1$, $C_y = 1$, $n = 1$ and $\partial F_1/\partial\sigma_y = 0$ and with equations (35) and (36), gives respectively:

$$d\varepsilon_y^p = d\lambda \quad \text{and} \quad A = H$$

If the same criterion were written as:

$$F = \bar{\sigma}^2 - \tfrac{1}{3}\sigma^2 = 0 \tag{41}$$

then, with $\alpha = 1$, $C_y = \frac{1}{3}$, $n = 2$ and $\partial F_1/\partial\sigma = 0$ the following would be obtained with equations (35) and (36):

$$d\varepsilon_y^p = \tfrac{2}{3}\, d\lambda\sigma_y \quad \text{and} \quad A = \tfrac{4}{9}H\sigma_y^2$$

The way in which a particular yield criterion is formulated will of course have no effect on the stress-strain relation $[S_{ep}]$ ultimately obtained, nor on the magnitude of $d\varepsilon_y^p$. The reason is that the manner of formulation also affects equation (17) with which the factor $d\lambda$ occurring in the various formulas is determined.

4 Hardening formulas

4.1 *Introduction*

The degree of agreement between the calculated and the actual material behaviour depends to a great extent on the hardening diagram introduced into the hardening model. This diagram will be considered in more detail in the present chapter. First, it will be examined how such a diagram could be determined with the aid of tests, more particularly in Sections 4.2 and 4.3 for work hardening and strain hardening respectively. It will often be found necessary to distinguish between different hardening diagrams as functions of the state of stress (uniaxial, biaxial, triaxial, tension-tension, tension-compression, etc.). This causes the need for a method of interpolation between the values which follow from the various diagrams, an example of which is given in Sections 4.4 and 4.5.

4.2 *Work hardening*

A hardening diagram is often associated with a simple uniaxial test as envisaged in Fig. 3. Fig. 3a gives the σ-ε-diagram which is obtained on measuring the stresses and strains

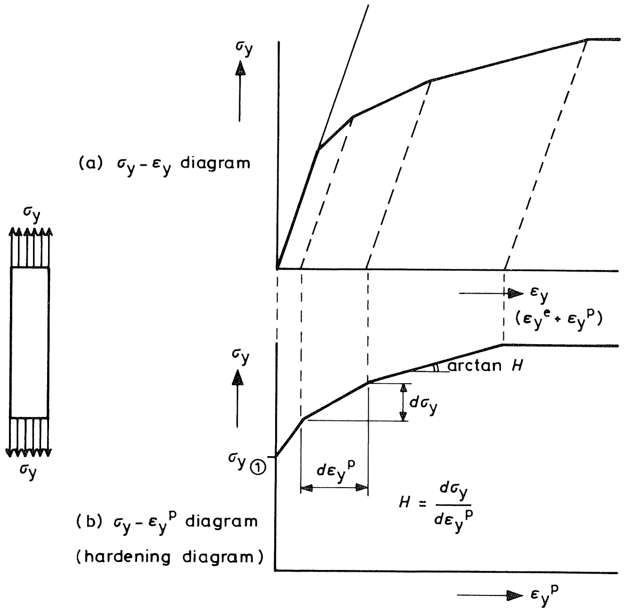


Fig. 3. Example of a hardening diagram and the relation with a σ-ε-diagram for a uniaxial specimen.

for a discrete number of load increments. The measured total strain can be split up into an elastic and a plastic portion, as follows:

$$\varepsilon_y = \varepsilon_y^e + \varepsilon_y^p \tag{42}$$

The magnitude of σ_y is measured for each load increment. Also it is known up to what strain ε_y the material is still just fully elastic (at the stress $\sigma_{y①}$ and the strain $\varepsilon_{y①}^e$), so that each load it is possible to calculate ε_y^e from:

$$\varepsilon_y^e = \frac{\sigma_y}{\sigma_{y①}} \varepsilon_{y①}^e \tag{43}$$

and so that ε_y^p can be determined from equation (42), whence:

$$\varepsilon_y^p = \varepsilon_y - \frac{\sigma_y}{\sigma_{y①}} \cdot \varepsilon_{y①}^e \tag{44}$$

By plotting the measured σ_y against the associated ε_y^p for each load increment, we obtain the hardening diagram as presented in Fig. 3b. The slope of this diagram is characterized by the hardening parameter H, which varies per load increment. This can be written as follows:

$$\Delta\sigma_y = H\,\Delta\varepsilon_y^p \tag{45}$$

which has been used in equation (22).

This adequately explains the uniaxial case. In the case of a specimen under multi-axial load it is still possible to base oneself on a uniaxial hardening diagram. The stress σ_y is now not measured directly, but is a comparison stress, which is calculated by substitution of the measured values of the stresses $\{\sigma\}$ into the yield function, whence σ_y is obtained. The calculation of the associated value of ε_y^p is also a little more complicated than in the uniaxial case. For this purpose we first determine, for the measured values of $\{\sigma\}$ and $\{\varepsilon\}$, the associated plastic strains $\{\varepsilon^p\}$ as follows:

$$\{\varepsilon^p\} = \{\varepsilon\} - \left\{\frac{\sigma}{\sigma_{①}} \cdot \varepsilon^e_{①}\right\} \tag{46}$$

where $\{\sigma_{①}\}$ and $\{\varepsilon^e_{①}\}$ are stresses and strains at the borderline between the elastic and the plastic range. Now that $\{\varepsilon^p\}$ is known for each increment, the increase in dissipated energy per load increment can be calculated from:

$$\Delta W_p = \{\Delta \varepsilon_y^p\}^T \{\sigma\} \tag{47}$$

Then, finally, $\Delta \varepsilon_y^p$ is obtained from:

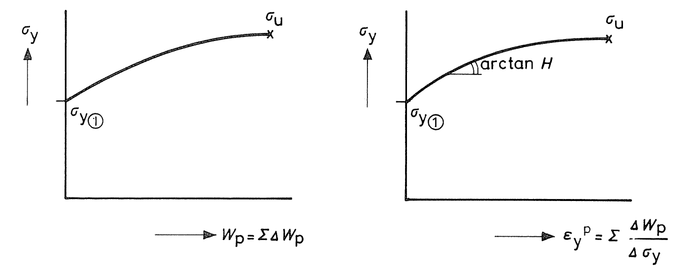
$$\Delta \varepsilon_y^p = \frac{\Delta W_p}{\Delta \sigma_y} \tag{48}$$

On plotting σ_y against $\Sigma \Delta \varepsilon_y^p$ we obtain the hardening diagram in the form of uniaxial $\sigma_y - \varepsilon_y^p$ diagram (Fig. 4b). There is, incidentally, no need to convert to a $\sigma_y - \varepsilon_y^p$ diagram. It is alternatively possible directly to work with a $\sigma_y - W_p$ diagram in which σ_y has been plotted against $\Sigma \Delta W_p$ (Fig. 4a).

4.3 *Strain hardening*

An equivalent $\sigma_y - \varepsilon_y^p$ diagram is used in the case of strain hardening too. Compiling this diagram is based on the plastic strain invariant $d\bar{\varepsilon}_p$, for which we have:

$$d\bar{\varepsilon}_p = \sqrt{d\varepsilon_{ij}\, d\varepsilon_{ij}} \tag{49}$$



a) $\sigma_y - W_p$ diagram for an arbitrary loading case

b) transformation into a uniaxial $\sigma_y - \varepsilon_y^p$ diagram

Fig. 4.

For materials possessing volume stability, such as steel, it is convenient to define $d\varepsilon_y^p$ as follows:

$$d\varepsilon_y^p = d\bar{\varepsilon}_p \sqrt{\tfrac{2}{3}} \tag{50}$$

In this way, in a uniaxial test the calculated value of $d\varepsilon_y^p$ will correspond to the measured value, because for such a test on a volume-stable material the following relation holds:

$$d\varepsilon_{22}^p = d\varepsilon_{33}^p = -0{,}5\ d\varepsilon_{11}^p$$

For $d\bar{\varepsilon}_p$ we thus obtain:

$$\delta\bar{\varepsilon}_p^2 = (1^2 + 0{,}5^2 + 0{,}5^2)\ (d\varepsilon_{11}^p)^2 = \tfrac{3}{2}(d\varepsilon_{11}^p)^2$$

Hence the factor $\frac{2}{3}$ is needed to ensure the validity of $d\varepsilon_y^p = d\varepsilon_{11}^p$. For materials which do not possess volume stability this factor $\frac{2}{3}$ does not arise, but equation (50) can nevertheless still be applied.

The calculation of $d\varepsilon_y^p$ presents no further problem. As has already been described with reference to work hardening, σ_y and ε_{ij}^p can, for each load increment, be determined from the measured data. With the aid of equation (50) the equivalent plastic strain increment $\Delta\varepsilon_y^p$ can be calculated as follows:

$$\Delta\varepsilon_y^p = \sqrt{\tfrac{2}{3}\ \Delta\varepsilon_{ij}^p\ \Delta\varepsilon_{ij}^p}$$

σ_y can then again be plotted against $\Sigma\Delta\varepsilon_y^p$.

Now that both modes of hardening have been discussed, it will be evident that the shape of the hardening diagram for one and the same material depends on whether strain hardening or work hardening is applied. Only in the exceptional case of a uniaxial test on a volume stable material (yield criterion of Von Mises or of Tresca) can the same hardening diagram be adopted in the calculation for both types of hardening.

It should also be pointed out that in the case of work hardening it is sometimes possible to manage with fewer measurements than in the case of strain hardening. This follows from equation (47), from which it appears that if a particular stress component is always zero during the test, the corresponding strain need not be measured, since no contribution is made to ΔW_p in the direction under consideration anyway.

From the foregoing it will have emerged that the shape of the hardening diagram depends not only on the type of material, but also on:

- the stress combination (uniaxial, biaxial, etc.)
- the hardening model (work hardening, strain hardening)
- the yield criterion applied (for it is with this that the magnitude of σ_y is calculated).

4.4 *Interpolation between arbitrary hardening diagrams*

As already stated, the shape of the hardening diagram depends on the stress combination that occurs (uniaxial, biaxial, etc.). Since it is, in a given analysis, often not known in advance what kind of stress combination will occur, it may be useful to apply the interpolation procedure that will now be described. It will be explained with reference

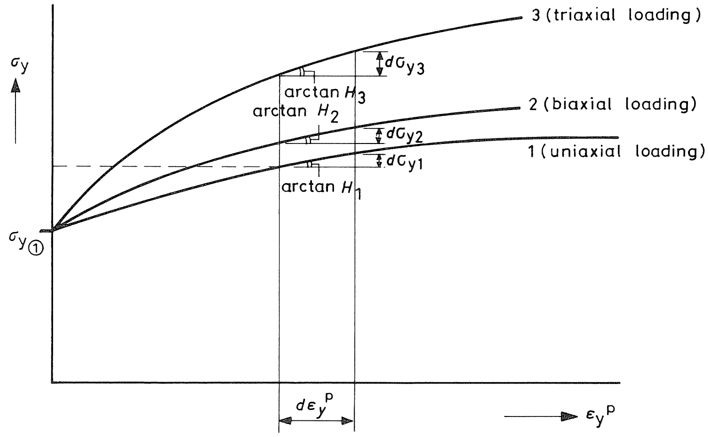


Fig. 5. Interpolation between different hardening diagrams.

to Fig. 5, which shows three hardening diagrams (numbered 1, 2 and 3) relating to uniaxial, biaxial and triaxial loading respectively. For the sake of convenience, since we are here concerned only with the principle involved, it is assumed that the shape of these diagrams is independent of whether there is tension, compression or a combination of the two.

In general, any particular load combination will have to be situated somewhere between these diagrams. It is now necessary to make a judicious choice of a distribution rule for interpolating between the available hardening diagrams.

Examples of the choice of such a distribution rule are given in Appendix B.

For the example in Fig. 5 the choice of the distribution rule comes to writing the increase in magnitude of the equivalent stress σ_y as follows:

$$d\sigma_y = k_1\, d\sigma_{y1} + k_2\, d\sigma_{y2} + k_3\, d\sigma_{y3} \tag{51}$$

while the distribution rule is:

$$k_1 + k_2 + k_3 = 1 \tag{52}$$

Equation (51) can now be written as:

$$d\sigma_y = (k_1 H_1 + k_2 H_2 + k_3 H_3)\, d\varepsilon_y^p \tag{53}$$

with reference to equation (22):

$$d\sigma_y = H\, d\varepsilon_y^p \tag{22}$$

it follows that H can now be defined as:

$$H = k_1 H_1 + k_2 H_2 + k_3 H_3 \tag{54}$$

With this example presented in Fig. 5 the principle of interpolation will have been adequately illustrated.

To conclude this section the general formulation will be given. Suppose that there are n hardening diagrams and that a distribution rule has been established as follows:

$$\sum_{i=1}^{n} k_i = 1 \tag{55}$$

then the hardening modulus is obtained from:

$$H = \sum_{i=1}^{n} k_i H_i \tag{56}$$

The value of H_i is read in the i^{th} hardening diagram for the total strain ε_y^p attained at that instant.

4.5 *Interpolation between affine hardening diagrams*

For a material such as concrete the shape of the hardening diagram depends on the state of stress that occurs. This can be conceived quite easily by considering the uniaxial state of stress. The σ-ε-diagram for compression is very different from that for tension. However, as an acceptable approximation, the two diagrams can be regarded as affine, i.e., similar in shape, linked by a constant multiplication factor α (see Fig. 6a).

This same factor also links the hardening diagrams derivable from the σ-ε-diagrams (see Fig. 6b). Now if we assume, for any given loading condition, that there still exists

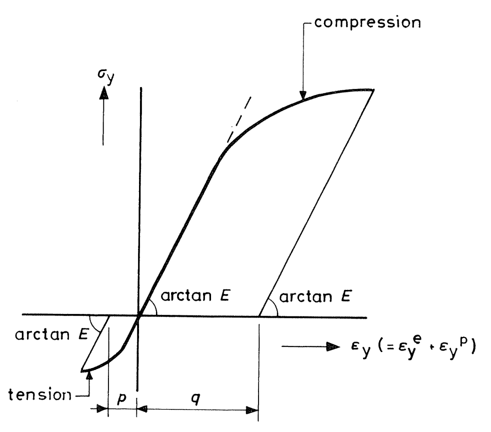


fig. 6a uniaxial σ–ε diagram

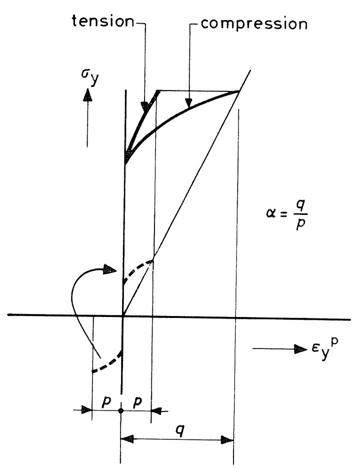


fig. 6b hardening diagrams belonging to fig. 6a

Fig. 6.

similarity of the diagram associated with uniaxial compression as the "basic diagram" (denoted by the subscript 1), then for any particular (arbitrary) diagram (i) we have:

$$\varepsilon_{yi}^{p} = \varepsilon_{y1}^{p} / \alpha_i \tag{57}$$

$$\mathrm{d}\varepsilon_{yi}^{p} = \mathrm{d}\varepsilon_{y1}^{p} / \alpha_i \tag{58}$$

$$H_i = \alpha_i H_1 \tag{59}$$

where the subscript i relates to the hardening diagram i. The further application of the procedure is illustrated in Fig. 7. The behaviour of α_i as a function of the stress combination associated with the diagram i is assumed to be known (see Appendix C). From the magnitude of α_i it will be apparent which of the hardening diagrams is actually applicable at any given instant. Since the value of α_i is continually changing in the calculation process, it means really that a different hardening diagram is being applied all the time. It is possible, however, always to go on working with the same basic diagram (diagram 1). For further clarification, assume for example that for a particular step in the calculation the diagram 2 (see Fig. 7) is the relevant hardening diagram. For the total strain ε_{y2}^{p} which occurs we then find the corresponding slope H_2.

If we wish to work only with diagram 1, the procedure is that we must read the slope in this diagram, not at ε_{y2}^{p}, but at $\varepsilon_{y1}^{p} = \alpha_2 \varepsilon_{y2}^{p}$. We thus read the slope H_1 for which $H_1 = H_2/\alpha_2$.

Therefore, the magnitude of $\mathrm{d}\sigma$ in equation (22) remains unchanged, since it can readily be seen that $\mathrm{d}\sigma_y = H_2\ \mathrm{d}\varepsilon_{y2}^{p} = H_1\ \mathrm{d}\varepsilon_{y1}^{p}$.

The above-mentioned procedure affects the derivation of the hardening formulas. In the case where hardening is conceived as dependent on W_p this means that in equation (34), in Section 3.3 the coefficient α occurs in the denominator:

$$\mathrm{d}W_p = \frac{1}{\alpha}\sigma_y\ \mathrm{d}\varepsilon_y^{p} \tag{34}$$

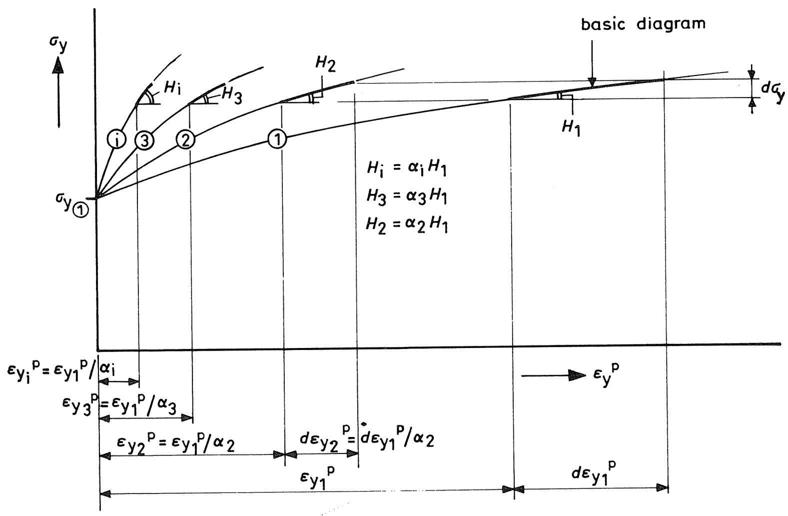


Fig. 7. Affine hardening diagrams.

This is so because the actual strain increment is smaller by a factor $1/\alpha$ than the increment in the basic diagram in which the value associated with H is read. For the same reason the coefficient occurs in equation (24) in the case of strain hardening (Section 3.2), namely:

$$d\varepsilon_y^p = \alpha \sqrt{\tfrac{2}{3}\, d\varepsilon_{ij}^p\, d\varepsilon_{ij}^p} \qquad (24)$$

Finally, it is to be noted that in a case where no interpolation at all is required because there is only one hardening diagram (applicable to all possible load combinations) to be considered, or where interpolation is done in accordance with Section 4.4, the value to be adopted for this coefficient is $\alpha = 1$.

5 Concluding remarks

In this article is has been explained how the phenomenon of hardening can be incorporated into the elasto-plastic mathematical model. A distinction is drawn between work hardening and strain hardening. Two parameters are found to be essential for the purpose, namely, the equivalent plastic strain increment $d\varepsilon_y^p$ and the hardening parameter A. It proves possible to establish for this hardening parameter a general formula (equation 23) which is independent of what hardening model and what yield criterion are adopted.

The formulas for the determination of $d\varepsilon_y^p$ and A have been further developed in Appendix A for a number of yield criteria and summarized in Table 1.

The advantage of the formulas given in this article lies more particularly in their general applicability. Because of this, if a new yield criterion is introduced, the associated formulation of $d\varepsilon_y^p$ and A presents no problem at all. If, instead of one hardening diagram, a number of hardening diagrams as functions of the state of stress are introduced, an improvement of the results can be obtained with the aid of the described interpolation methods.

APPENDIX A

Further development of hardening formulas

A1 General

The formulas for $d\varepsilon_y^p$ and A for the yield criteria of Tresca, Von Mises, Buyukozturk, Drucker-Prager and Mohr-Coulomb will be elaborated here.

For Buyukozturk's criterion the formulation indicated in [7] will be worked out, but also the variant envisaged in Appendix D.

For the other criteria the starting point will be the formulation employed by Zienkiewicz in [5], which expresses the yield criteria in the stress invariants σ_{okt}, $\bar{\sigma}$ and Φ. The yield criteria are summarized in Table A1.

The quantities $\bar{\sigma}$ and Φ are represented, for a given value, in a so-called deviatoric section in Fig. A1. The material constants c (cohesion) and θ (friction) associated with the Drucker-Prager and Mohr-Coulomb criteria in the equations (A15) and (A19) can be expressed as functions of the equivalent yield stress σ_y, as follows:

$$\sin\,\theta = \frac{\sigma_y - f_t}{\sigma_y + f_t} \tag{A1}$$

$$c = \tfrac{1}{2}\sigma_y \frac{1 - \sin\,\theta}{\cos\,\theta} \tag{A2}$$

where f_t denotes the uniaxial tensile strength.

For some yield criteria it will, for the further derivation, be more convenient to base oneself, not on the formulation given in Table 1, but on a formulation in terms of principle stresses σ_1, σ_2 and σ_3. It is then furthermore necessary to express the function always in σ_y and not, as in the case of Mohr-Coulomb and Drucker-Prager, in the cohe-

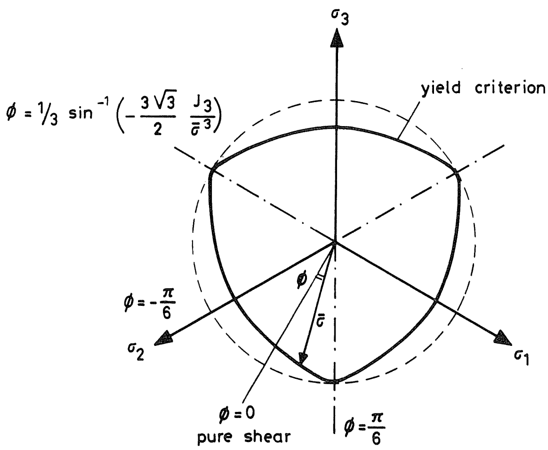


Fig. A1. Deviatoric section.

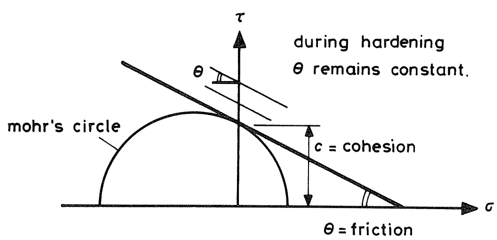


Fig. A2. Mohr Coulomb yield criterion.

sion c (see equation (A15) and (A16)). Because of this the hardening formulas derived in Chapter 3 cannot be applied.

The cohesion c can, however, be got rid of with the aid of equation (A2): a yield criterion expressed in σ_y and θ is thus obtained. Since isotropic hardening is presumed, the friction θ can be taken to remain constant during hardening, so that it causes no trouble in the process of differentiation, with regard to the constancy of θ see, for example, Fig. A2 relating to the Mohr-Coulomb yield criterion.

A2 Work hardening

TRESCA:

The formulation given in Table 1:

$$F = 2\bar{\sigma}\cos\Phi - \sigma_y = 0 \tag{A3}$$

is replaced by:

$$F = \underbrace{\sigma_1 - \sigma_3}_{F_1} - \sigma_y = 0 \tag{A4}$$

Since $\partial F_1/\partial\sigma_y = 0$, while $n = 1$ and $C_y = 1$, it follows from equations (35) and (36) that:

$$\mathrm{d}\varepsilon_y^p = \alpha\,\mathrm{d}\lambda \tag{A5}$$

$$A = \alpha H \tag{A6}$$

VON MISES:

The formulation is:

$$\underbrace{F = \bar{\sigma}\sqrt{3}}_{F_1} - \sigma_y = 0 \tag{A7}$$

With $\partial F_1/\partial\sigma_y = 0$ and $n = 1$ and $C_y = 1$ we again obtain from equations (35) and (36):

$$\mathrm{d}\varepsilon_y^p = \alpha\,\mathrm{d}\lambda \tag{A5}$$

$$A = \alpha H \tag{A6}$$

BUYUKOZTURK [7]:

$$F = 3\underbrace{\sqrt{3\bar{\sigma}^2 + 3\sigma_y\sigma_{\text{okt}} + \tfrac{9}{5}\sigma_{\text{okt}}}}_{F_1} - \sigma_y = 0 \tag{A8}$$

This is a homogeneous function of degree $n = 1$, while $C_y = 1$.

Furthermore, for $\partial F_1/\partial\sigma_y$:

$$\frac{\partial F_1}{\partial\sigma_y} = \frac{9\sigma_{\text{okt}}}{2\sqrt{3\bar{\sigma}^2 + 3\sigma_y\sigma_{\text{okt}} + \tfrac{9}{5}\sigma_{\text{okt}}^2}} = \frac{27\sigma_{\text{okt}}}{2\sigma_y} \tag{A9}$$

From equations (35) and (36) we now obtain:

$$\mathrm{d}\varepsilon_y^p = \alpha\,\mathrm{d}\lambda\left(1 - \frac{27\sigma_{\mathrm{okt}}}{2\sigma_y}\right) \tag{A10}$$

$$A = \alpha H\left(1 - \frac{27\sigma_{\mathrm{okt}}}{2\sigma_y}\right)^2 \tag{A11}$$

BUYUKOZTURK (variant, see Appendix D):

$$F = \underbrace{27\bar{\sigma}^2 + 27\sigma_y\sigma_{\mathrm{okt}} + \tfrac{81}{5}\sigma_{\mathrm{okt}}^2}_{F_1} - \sigma_y^2 = 0 \tag{A12}$$

For $\partial F_1/\partial\sigma_y$ we obtain:

$$\frac{\partial F_1}{\partial\sigma_y} = 27\sigma_{\mathrm{okt}}$$

The function F is of degree $n=2$, while again $C_y=1$.
Now from equations (35) and (36):

$$\mathrm{d}\varepsilon_y^p = \alpha\,\mathrm{d}\lambda(2\sigma_y - 27\sigma_{\mathrm{okt}}) \tag{A13}$$

$$A = \alpha H(2\sigma_y - 27\sigma_{\mathrm{okt}})^2 \tag{A14}$$

DRUCKER-PRAGER:

$$F = \frac{6\sigma_{\mathrm{okt}}\sin\theta}{(3-\sin\theta)\sqrt{3}} + \bar{\sigma} + \frac{6c\cos\theta}{(3-\sin\theta)\sqrt{3}} = 0 \tag{A15}$$

With equation (A2) this can alternatively be written as follows:

$$F = \underbrace{\frac{6\sigma_{\mathrm{okt}}\sin\theta}{(3-\sin\theta)\sqrt{3}}}_{F_1} + \bar{\sigma} - \sigma_y\frac{(1-\sin\theta)}{3-\sin\theta}\sqrt{3} = 0 \tag{A16}$$

With $\dfrac{\partial F_1}{\partial\sigma_y} = 0$, $n=1$ and $C_y = \dfrac{(1-\sin\theta)}{3-\sin\theta}\sqrt{3}$

we obtain from equations (35) and (36):

$$\mathrm{d}\varepsilon_y^p = \alpha\,\mathrm{d}\lambda\frac{(1-\sin\theta)}{3-\sin\theta}\sqrt{3} \tag{A17}$$

$$A = \frac{3\alpha H(1-\sin\theta)}{3-\sin\theta} \tag{A18}$$

MOHR-COULOMB:

$$F = \sigma_{\text{okt}} \sin\theta + \bar{\sigma}\cos\Phi - \frac{\bar{\sigma}}{\sqrt{3}} \sin\varphi \sin\theta - c\cos\theta = 0 \tag{A19}$$

On substitution of the formulas expressing σ_{okt}, $\bar{\sigma}$ and Φ in terms of principal stresses (see notation), equation (A19) can alternatively be written as follows:

$$F = \tfrac{1}{2}(\sigma_1 - \sigma_3) - c\cos\theta + \tfrac{1}{2}(\sigma_1 + \sigma_3)\sin\theta = 0 \tag{A20}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$

On substitution of equation (A2) we obtain:

$$F = \underbrace{\tfrac{1}{2}(\sigma_1 - \sigma_3) + \tfrac{1}{2}(\sigma_1 + \sigma_3)\sin\theta}_{F_1} - \tfrac{1}{2}(1 - \sin\theta)\sigma_y = 0 \tag{A21}$$

With $\partial F_1/\partial\sigma_y = 0$, $n = 1$ and $C_y = \frac{1}{2}(1 - \sin\theta)$ we obtain from equations (35) and (36):

$$\mathrm{d}\varepsilon_y^p = \alpha\,\mathrm{d}\lambda \left(\frac{1 - \sin\theta}{2}\right) \tag{A22}$$

$$A = \alpha H \left(\frac{1 - \sin\theta}{2}\right)^2 \tag{A23}$$

A3 Strain hardening

TRESCA:

The formulation given in equation (A4) is the starting point:

$$F = \underbrace{\sigma_1 - \sigma_3}_{F_1} - \sigma_y = 0 \tag{A4}$$

Since F_1 is expressed in terms of principal stresses, we can write:

$$\frac{\partial F_1}{\partial \sigma_{ij}} \cdot \frac{\partial F_1}{\partial \sigma_{ij}} = \frac{\partial F_1}{\partial \sigma_1} \cdot \frac{\partial F_1}{\partial \sigma_1} + \frac{\partial F_1}{\partial \sigma_2} \cdot \frac{\partial F_1}{\partial \sigma_2} + \frac{\partial F_1}{\partial \sigma_3} \cdot \frac{\partial F_1}{\partial \sigma_3} = 1.1 + 0.0 + (-1)\cdot(-1) = 2$$

so that:

$$\sqrt{\frac{2}{3} \frac{\partial F_1}{\partial \sigma_{ij}} \frac{\partial F_1}{\partial \sigma_{ij}}} = \frac{2}{3}\sqrt{3}$$

With $n = 1$ and $C_y = 1$ and $\partial F_1/\partial\sigma_y = 0$ we obtain from equations (26) and (27)

$$\mathrm{d}\varepsilon_y^p = \tfrac{2}{3}\alpha\,\mathrm{d}\lambda\sqrt{3} \tag{A24}$$

$$A = \tfrac{2}{3}\alpha H\sqrt{3} \tag{A25}$$

VON MISES:

The formulation given in equation (A7) is:

$$F=\underbrace{\bar{\sigma}\sqrt{3}}_{F_1}-\sigma_y=0$$

For determining $\partial F_1/\partial\sigma_y$ it should be borne in mind that:

$$\bar{\sigma}=\sqrt{\tfrac{1}{6}\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\}} \qquad \text{(see notation)}$$

whence can be deduced:

$$\frac{\partial F_1}{\partial\sigma_1}=\frac{\partial\bar{\sigma}}{\partial\sigma_1}\sqrt{3}=\left(\frac{\sigma_1-\sigma_{\mathrm{okt}}}{2\bar{\sigma}}\right)\sqrt{3}$$

$$\frac{\partial F_2}{\partial\sigma_2}=\frac{\partial\bar{\sigma}}{\partial\sigma_2}\sqrt{3}=\left(\frac{\sigma_2-\sigma_{\mathrm{okt}}}{2\bar{\sigma}}\right)\sqrt{3}$$

$$\frac{\partial F_3}{\partial\sigma_3}=\frac{\partial\bar{\sigma}}{\partial\sigma_3}\sqrt{3}=\left(\frac{\sigma_3-\sigma_{\mathrm{okt}}}{2\bar{\sigma}}\right)\sqrt{3}$$

from which it follows that:

$$\sqrt{\frac{2}{3}\frac{\partial F_1}{\partial\sigma_{ij}}\frac{\partial F_1}{\partial\sigma_{ij}}}=\sqrt{\frac{2}{3}\left[\left(\frac{\partial F_1}{\partial\sigma_1}\right)^2+\left(\frac{\partial F_1}{\partial\sigma_2}\right)^2+\left(\frac{\partial F_1}{\partial\sigma_3}\right)^2\right]}$$

from which we obtain, on rearrangement:

$$\sqrt{\frac{2}{3}\frac{\partial F_1}{\partial\sigma_{ij}}\frac{\partial F_1}{\partial\sigma_{ij}}}=1$$

Equations (26) and (27), with $\partial F_1/\partial\sigma_y=0$, $n=1$, $C_y=1$ give the following expressions:

$$\mathrm{d}\varepsilon_y^p=\alpha\,\mathrm{d}\lambda \tag{A26}$$

$$A=\alpha H \tag{A27}$$

BUYUKOZTURK [7]:

$$F=\underbrace{3\sqrt{3\bar{\sigma}^2+3\sigma_y\sigma_{\mathrm{okt}}+\tfrac{9}{5}\sigma_{\mathrm{okt}}}}_{F_1}-\sigma_y=0 \tag{A8}$$

For $\partial F_1/\partial\sigma_y$ we have according to equation (A9):

$$\frac{\partial F_1}{\partial\sigma_y}=\frac{27\sigma_{\mathrm{okt}}}{2\sigma_y}$$

Furthermore: $n=1$ and $C_y=1$. Unfortunately, it is not possible further to simplify $\partial F_1/\partial\sigma_{ij}$; this term will have to be determined numerically. From equations (26) and (27) we obtain:

$$d\varepsilon_y^p = \alpha\, d\lambda \sqrt{\frac{2}{3}\frac{\partial F_1}{\partial\sigma_{ij}}\frac{\partial F_1}{\partial\sigma_{ij}}} \tag{A28}$$

$$A = \alpha H\left(1-\frac{27\sigma_{\text{okt}}}{2\sigma_{ij}}\right)\sqrt{\frac{2}{3}\frac{\partial F_1}{\partial\sigma_{ij}}\frac{\partial F_1}{\partial\sigma_{ij}}} \tag{A29}$$

BUYUKOZTURK (variant, see Appendix D)

$$F = \underbrace{27\bar{\sigma}^2 + 27\sigma_y\sigma_{\text{okt}} + \tfrac{81}{5}\sigma_{\text{okt}}^2}_{F_1} - \sigma_y^2 = 0 \tag{12}$$

Here again it is not possible to simplify $\partial F_1/\partial\sigma_{ij}$

With $n=2$, $C_y=1$ and $\partial F_1/\partial\sigma_y=0$ we obtain from equations (26) and (27)

$$d\varepsilon_y^p = \alpha\, d\lambda \sqrt{\frac{2}{3}\frac{\partial F_1}{\partial\sigma_{ij}}\frac{\partial F_1}{\partial\sigma_{ij}}} \tag{A30}$$

$$A = \alpha H\,(2\sigma_y - 27\sigma_{\text{okt}})\sqrt{\frac{2}{3}\frac{\partial F_1}{\partial\sigma_{ij}}\frac{\partial F_1}{\partial\sigma_{ij}}} \tag{A31}$$

DRUCKER-PRAGER:

The formulation employed in equation (A15) is as follows:

$$F = \underbrace{\frac{6\sigma_{\text{okt}}\sin\theta}{(3-\sin\theta)\sqrt{3}} + \bar{\sigma}}_{F_1} - \frac{\sigma_y(1-\sin\theta)}{3-\sin\theta} = 0 \tag{A15}$$

whence can be deduced:

$$\frac{\partial F_1}{\partial\sigma_1} = \frac{2\sin\theta}{(3-\sin\theta)}\sqrt{3} + \sigma_1 - \sigma_{\text{okt}}$$

$$\frac{\partial F_1}{\partial\sigma_2} = \frac{2\sin\theta}{(3-\sin\theta)}\sqrt{3} + \sigma_2 - \sigma_{\text{okt}}$$

$$\frac{\partial F_1}{\partial\sigma_3} = \frac{2\sin\theta}{(3-\sin\theta)}\sqrt{3} + \sigma_3 - \sigma_{\text{okt}}$$

from which it follows that:

$$\frac{\partial F_1}{\partial\sigma_{ij}}\cdot\frac{\partial F_1}{\partial\sigma_{ij}} = \left(\frac{\partial F_1}{\partial\sigma_1}\right)^2 + \left(\frac{\partial F_1}{\partial\sigma_2}\right)^2 + \left(\frac{\partial F_1}{\partial\sigma_3}\right)^2 = \frac{4\sin^2\theta}{(3-\sin\theta)^2} + \frac{1}{2}$$

so that:

$$\frac{2}{3}\sqrt{\frac{\partial F_1}{\partial\sigma_{ij}}\frac{\partial F_1}{\partial\sigma_{ij}}}=\sqrt{\frac{8\sin^2\theta}{3(3-\sin\theta)^2}+\tfrac{1}{3}}$$

with $n=1$, $\frac{\partial F_1}{\partial\sigma_y}=0$ and $C_y=\frac{1-\sin\theta}{3-\sin\theta}$ we obtain from equations (26) and (27):

$$\mathrm{d}\varepsilon_y^p=\mathrm{d}\lambda\sqrt{\frac{8\sin^2\theta}{3(3-\sin\theta)^2}+\tfrac{1}{3}} \qquad \text{(A32)}$$

$$A=H\frac{(1-\sin\theta)}{(3-\sin\theta)}\sqrt{3}\sqrt{\frac{8\sin^2\theta}{3(3-\sin\theta)^2}+\tfrac{1}{3}} \qquad \text{(A33)}$$

If θ is allowed to approach zero, it follows that: $\mathrm{d}\varepsilon_y^p=\frac{1}{3}\alpha\,\mathrm{d}\lambda\sqrt{3}$ and $A=\frac{1}{3}\alpha H\sqrt{3}$. This apparently does not tally with the corresponding values for the Von Mises method, for which, according to equations (A26) and (A27), we obtained: $\mathrm{d}\varepsilon_y^p=\alpha\,\mathrm{d}\lambda$ and $A=\alpha H$.

The difference is attributable to the fact, that when θ in equation (A15) approaches zero, the yield function becomes:

$$F=\underbrace{\bar{\sigma}}_{F_1}-\tfrac{1}{3}\sigma_y\sqrt{3}=0 \qquad \text{(A34)}$$

which is not in agreement with the formulation of equation (A7).

For

$$\sqrt{\frac{2}{3}\frac{\partial F_1}{\partial\sigma_{ij}}\frac{\partial F_1}{\partial\sigma_{ij}}}$$

we can deduce for the formulation of equation (A34)

$$\sqrt{\frac{2}{3}\frac{\partial F_1}{\partial\sigma_{ij}}\frac{\partial F_1}{\partial\sigma_{ij}}}=\tfrac{1}{3}\sqrt{3}$$

(see the derivation for Von Mises on page 30).

With $n=1$,

$$\frac{\partial F_1}{\partial\sigma_{ij}}=0 \quad \text{and} \quad C_y=\tfrac{1}{3}\sqrt{3}$$

this gives, on substitution into equations (26) and (27):

$$\mathrm{d}\varepsilon_y^p=\tfrac{1}{3}\,\mathrm{d}\lambda\sqrt{3} \qquad \text{(A35)}$$

$$A=\tfrac{1}{3}H\sqrt{3} \qquad \text{(A36)}$$

which is also obtained from equation (A32) and (A33) by allowing θ in them to approach zero.

MOHR-COULOMB:

The formulation given in equation (A21) forms the starting point:

$$F = \underbrace{\tfrac{1}{2}(\sigma_1 - \sigma_3) + \tfrac{1}{2}(\sigma_1 + \sigma_3)\sin\theta}_{F_1} - \tfrac{1}{2}(1 - \sin\theta)\sigma_y = 0 \qquad \text{(A21)}$$

$$\frac{\partial F_1}{\partial \sigma_1} = \tfrac{1}{2}(1 + \sin\theta)$$

$$\frac{\partial F_1}{\partial \sigma_2} = 0$$

$$\frac{\partial F_1}{\partial \sigma_3} = \tfrac{1}{2}(1 - \sin\theta)$$

Hence we obtain:

$$\frac{\partial F_1}{\partial \sigma_{ij}}\frac{\partial F_1}{\partial \sigma_{ij}} = \left(\frac{\partial F_1}{\partial \sigma_1}\right)^2 + \left(\frac{\partial F_1}{\partial \sigma_2}\right)^2 + \left(\frac{\partial F_1}{\partial \sigma_3}\right)^2 =$$

$$\tfrac{1}{4}(1 + \sin\theta)^2 + (0)^2 + \tfrac{1}{4}(1 - \sin\theta)^2 = \tfrac{1}{2}(1 + \sin^2\theta)$$

so that:

$$\sqrt{\frac{2}{3}\frac{\partial F_1}{\partial \sigma_{ij}}\frac{\partial F_1}{\partial \sigma_{ij}}} = \sqrt{\frac{1 + \sin^2\theta}{3}}$$

With $n = 1$, $\partial F_1/\partial \sigma_y = 0$ and $C_y = \tfrac{1}{2}(1 - \sin\theta)$ equations (26) and (27) give:

$$d\varepsilon_y^p = d\lambda \sqrt{\frac{1 + \sin^2\theta}{3}} \qquad \text{(A37)}$$

$$A = \tfrac{1}{2}H(1 - \sin\theta)\sqrt{\frac{1 + \sin^2\theta}{3}} \qquad \text{(A38)}$$

APPENDIX B

Two examples of possible distribution rules for interpolation

B1 General

A method for carrying out interpolations between various hardening diagrams has been discussed in Section 4.2. This method requires the use of a distribution rule conforming to:

$$\sum_{i=1}^{n} k_i = 1 \qquad \text{(B1)}$$

The object is thereby to determine an equivalent hardening modulus H, as follows:

$$H = \sum_{i=1}^{n} k_i H_i \qquad \text{(B2)}$$

Two possible distribution rules will be presented here. The first relates to the case of three hardening diagrams relating to uniaxial, biaxial and triaxial loading respectively (whether tension, compression or a combination of the two occurs is immaterial). The second relates to the rather unlikely case where there are nine hardening diagrams available for all conceivable load combinations.

B2 Case 1

As already noted, there are three hardening diagrams available in this case. In Table B1 these are symbolized for uniaxial, biaxial and triaxial loading, designated as types, 1, 2 and 3 respectively. On the basis of Lagrange's interpolation polynomial it is indicated, in the same table, how the associated factors k_1, k_2 and k_3 can be calculated as functions

Table B1. Interpolation between three types of tests.

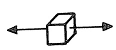
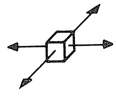
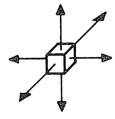
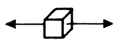
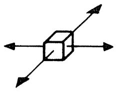
type 1		$k_1 = 1 - \dfrac{\sigma_2^2 + \sigma_3^2}{\sigma_1^2} + \dfrac{\sigma_2^2\sigma_3^2}{\sigma_1^4}$
type 2		$k_2 = \dfrac{\sigma_2^2 + \sigma_3^2}{\sigma_1^2} - 2\,\dfrac{\sigma_2^2\sigma_3^2}{\sigma_1^4}$
type 3		$k_3 = \dfrac{\sigma_2^2\sigma_3^2}{\sigma_1^4}$

Table B2. Interpolation between two types of tests.

type 1		$k_1 = 1 - \frac{\sigma_2^2 + \sigma_3^2}{\sigma_1^2} + \frac{\sigma_2^2\sigma_3^2}{\sigma_1^4}$
type 2		$k_2 = \frac{\sigma_2^2 + \sigma_3^2}{\sigma_1^2} - \frac{\sigma_2^2\sigma_3^2}{\sigma_1^4}$

of the principal stresses σ_1, σ_2 and σ_3. Since it is assumed to be of no importance whether compression, tension or a combination thereof occurs, the stresses σ_1, σ_2 and σ_3 occur quadratically in the formulas. The principal stresses must be arranged as follows:

$$\sigma_1^2 > \sigma_2^2 > \sigma_3^2$$

since the terms are divided by σ_1^2. In the case of hardening, σ_1^2 will never be zero, for this is possible only if the point under consideration is stressless. The factors, k_1, k_2 and k_3 do indeed fulfil the condition $k_1 + k_2 + k_3 = 1$, while the check shows that, if the purely uniaxial, biaxial or triaxial test is considered, the relevant value of k is in fact equal to 1 and the two others are both equal to 0.

In a case where only uniaxial and biaxial tests are available, Table B1 can be simplified to Table B2 by adding the coefficients k_2 and k_3. Then, if we can still be certain that σ_3 is always zero, we obtain:

$$k_1 = 1 - \frac{\sigma_2^2}{\sigma_1^2} \quad \text{and} \quad k_2 = \frac{\sigma_2^2}{\sigma_1^2}$$

B3 Case 2

Starting from the arrangement:

$$\sigma_1 > \sigma_2 > \sigma_3 \tag{B3}$$

We can use Table B3 for interpolation between nine types of tests which are indicated symbolically in that table.

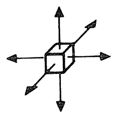
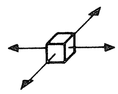
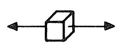
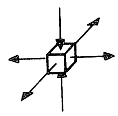
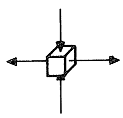
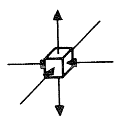
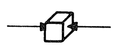
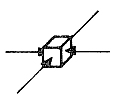
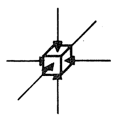
For the factors k_i the condition is again $\sum_{i=1}^{9} k_i = 1$.

In the formulas the stress σ_4 occurs in the denominator; it is defined as follows:

$$\sigma_4 = \tfrac{1}{2}(\sigma_1 - \sigma_3) + \tfrac{1}{2}|(\sigma_1 + \sigma_3)| \tag{B4}$$

This stress never becomes zero in the case of hardening, since equation (B4) can become zero only if $\sigma_1 = \sigma_3 = 0$. According to equation (B3), σ_2 must then also be zero, so that the point is stressless in that case.

Table B3. Interpolation between nine types of tests.

type of test					interpolation coefficient
1	+	+	+		$k_1 = \frac{1}{2}\frac{\sigma_3(\sigma_4+\sigma_3)}{\sigma_4^2} + 0{,}02C$
2	+	+	0		$k_2 = \frac{(\sigma_4+\sigma_3)(\sigma_2-\sigma_3)}{\sigma_4^2} + 0{,}12C$
3	+	0	0		$k_3 = \frac{(\sigma_4+\sigma_3)(\sigma_1-\sigma_2)}{\sigma_4^2} + 0{,}18C$
4	+	+	÷		$k_4 = \frac{1}{2}\frac{(\sigma_2-\sigma_3-\sigma_4)(\sigma_2-\sigma_3)}{\sigma_4^2} + 0{,}06C$
5	+	0	÷		$k_5 = \frac{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)}{\sigma_4^2} + 0{,}24C$
6	+	÷	÷		$k_6 = \frac{1}{2}\frac{(\sigma_1-\sigma_2-\sigma_4)(\sigma_1-\sigma_2)}{\sigma_4^2} + 0{,}06C$
7	0	0	÷		$k_7 = \frac{(\sigma_4-\sigma_1)(\sigma_2-\sigma_3)}{\sigma_4^2} + 0{,}18C$
8	0	÷	÷		$k_8 = \frac{(\sigma_4-\sigma_1)(\sigma_1-\sigma_2)}{\sigma_4^2} + 0{,}12C$
9	÷	÷	÷		$k_9 = \frac{1}{2}\frac{\sigma_1(\sigma_1-\sigma_4)}{\sigma_4^2} + 0{,}02C$

where: $C = \frac{(\sigma_4+\sigma_3)(\sigma_4-\sigma_1)}{\sigma_4^2}$; $\sigma_4 = \frac{1}{2}(\sigma_1-\sigma_3) + \frac{1}{2}|(\sigma_1-\sigma_3)|$

In practice, however, it is hardly conceivable that the associated hardening diagrams for all nine types of tests are known.

However, the table can still be of use, as will be shown with the aid of a simple example. Suppose that for type 5 there is no hardening diagram available, so that it is only possible to interpolate between eight types of tests. Now it can reasonably be assumed that the "missing" diagram is the average of the diagrams associated with the types 4 and 6. This assumption means that the formula for k_4 and k_6 most both be adjusted by the addition of half of k_5, so that:

$$k_4 = \tfrac{1}{2}\,\frac{(\sigma_1 - \sigma_3 - \sigma_4)(\sigma_2 - \sigma_3)}{\sigma_4^2} + 0{,}18C$$

$$k_6 = \tfrac{1}{2}\,\frac{(\sigma_1 - \sigma_3 - \sigma_4)(\sigma_2 - \sigma_3)}{\sigma_4^2} + 0{,}18C$$

The coefficient k_5 can now be omitted, with the result that eight of the nine coefficients remain.

In the manner outlined above it is always possible to compile, with the elements contained in Table B3, a new reduced table which can be used for the number of hardening diagrams actually available.

APPENDIX C

Example of an interpolation function for affine hardening diagrams

C1 General

In Section 4.3 an interpolation method is discussed in which, with the aid of a coefficient α, it is possible to interpolate between affine hardening diagrams. An example will now be given to show how such a function could be conceived.

The starting point is provided by the hardening diagram for uniaxial compression. For this chosen standard case we have $\alpha = 1$. Furthermore it will be endeavoured to tie this up as well as possible with the case of uniaxial tension and with the case of biaxial compression.

In order to represent the above-mentioned ranges in one diagram, we may consider a section through the yield surface along the hydrostatic axis (see Fig. C1). σ_{okt} has been plotted along the horizontal, and $\frac{1}{3}\bar{\sigma}\sqrt{3}$ along the vertical axis. The factor $\frac{1}{3}\sqrt{3}$ has been introduced in order to preserve the correct geometric ratio of the sections. It should be noted that the shape of the intersection varies in principle with the value of Φ (see Fig. A1), but since this is not of essential importance, only one section need be considered.

In this intersection it is indicated how the principal axes are located in relation to the hydrostatic axis, namely, at angles $\varphi = \frac{1}{4}\pi$ and $\frac{3}{4}\pi$. Lines for which $\sigma_1 = \sigma_2$ with $\sigma_3 = 0$, etc. are sloped at angles $\varphi = \frac{1}{8}\pi$ and $\frac{7}{8}\pi$.

Now it is assumed that a ratio of, for example, 1/10 exists between the σ-ε-diagram for uniaxial compression and the σ-ε for uniaxial tension. The ultimate tensile strain is therefore 1/10 of the ultimate compressive strain, while the corresponding tensile strength is equal to 1/10 of the compressive strength.

In the example it is further assumed that the biaxial compressive strength exceeds the uniaxial compressive strength by a factor of 1.2. This factor of 1.2 affects the whole σ-ε-diagram, so that the shape is the same as that for uniaxial compression with unchanged modulus of elasticity.

In the further treatment of the problem it is necessary to draw a distinction between strain hardening and work hardening.

C2 Strain hardening

The above-mentioned ratios 0.1 and 1.2 will also be more or less reflected in the hardening diagram, on the understanding that they relate only to the plastic strains ε_y^p, since the associated values of σ_y are now scaled, which is manifested, for example, in the fact that all the hardening diagrams begin with $\sigma_y = \sigma_{y①}$. Instead of continually using a different hardening diagram, we may introduce factors α, in accordance with Section 4.2, by means of which every possible hardening diagram is converted to the strain hardening diagram (in this case the diagram for uniaxial compression). The factor α for uniaxial

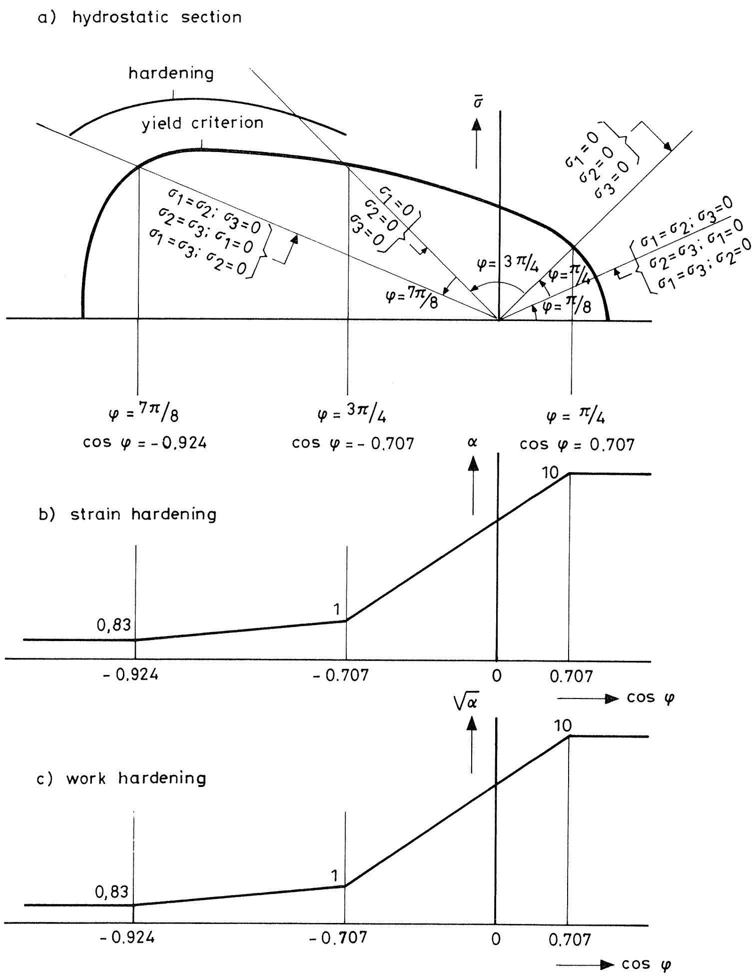


Fig. C1. Interpolation factor.

tension then becomes $1/0.1 = 10$ and that for biaxial compression becomes $1/1.2 = 0.83$.

The two cases are illustrated in Fig. C1, where the point $\alpha = 1$ is also shown. For convenience these respective points can be connected by straight lines, so that the variation pattern is known, except for ranges $\varphi < \pi/4$ and $\varphi > \frac{7}{8}\pi$. For these ranges, α may be assumed constant, namely $\alpha = 10$ and 0.83.

Of course, more complex interpolation formulas can be conceived. Considering the rather high degree of schematization arising from the assumption of affinity of the hardening diagrams, this is pointless, however.

The various ranges are demarcated with the aid of the angle φ. Arithmetically it is, however, more convenient to use cos φ. After each iterative step, cos φ is determinable from:

$$\cos \varphi = \frac{\sigma_m}{\sqrt{\sigma_{\text{okt}}^2 + \frac{1}{3}\bar{\sigma}^2}} \qquad \text{(C1)}$$

The following values correspond to the significant cases:

$$\cos 45° = 0.707 \qquad \cos 135° = -0.707 \qquad \cos 157\tfrac{1}{2}° = -0.924$$

With the aid of equation (C1) we now calculate the value of cos φ that arises. By linear interpolation with cos φ between the various known points the values for α are successively obtained:

$$\begin{aligned}
&\cos \varphi < -0.924 && \rightarrow \alpha = 0.83 \\
&-0.924 \leq \cos \varphi \leq -0.707 && \rightarrow \alpha = 0.92 + 0.13 \cos \varphi \\
&-0.707 \leq \cos \varphi \leq 0.707 && \rightarrow \alpha = 5.5 + 6.36 \cos \varphi \\
&\cos \varphi > 0.707 && \rightarrow \alpha = 10
\end{aligned}$$

These formulas describe the function presented in Fig. C1b, composed of straight lines.

C3 Work hardening

If hardening is taken as proportional to W_p, nothing changes in so far as the magnitude of α is concerned. Consider the case of uniaxial tension. In comparison with uniaxial compression, the stress as well as the strain are smaller by a factor of 0.1. Hence the dissipated energy will differ by a factor of 0.01. In both cases the degree of hardening is the same, however. Hence the factors α have to be squared. It would be possible to interpolate linearly between the squared values, but it is just as convenient to square the whole linear interpolation given in the preceding section, so that we obtain:

$$\begin{aligned}
&\cos \varphi < -0.924 && \rightarrow \alpha = 0.69 \\
&-0.924 \leq \cos \varphi \leq -0.707 && \rightarrow \alpha = (0.92 + 0.13 \cos \varphi)^2 \\
&-0.707 \leq \cos \varphi \leq -0.707 && \rightarrow \alpha = (5.5 + 6.36 \cos \varphi)^2 \\
&\cos \varphi > 0.707 && \rightarrow \alpha = 100
\end{aligned}$$

These formulas have been plotted in Fig. C1c, which is entirely similar to Fig. C1b, except that the ordinates are values of $\sqrt{\alpha}$ instead of α.

APPENDIX D

Variant formulation of Buyukozturk's criterion

Two formulations of Buyukozturk's yield criterion have been considered in this article. This clearly illustrates that the magnitude of A and $d\varepsilon_y^p$ depends on the way in which the criterion is formulated. In this appendix it will be shown why the variant formulation is preferable for practical calculations.

To clarify this, Buyukozturk's original formulation with a section along the hydrostatic axis is shown in Fig. D1.

Its shape turns out to be an ellipse, which comprises an imaginary region due to the fact that in the formulation:

$$F = 3\sqrt{3\bar{\sigma}^2 + 3\sigma_y\sigma_{\text{okt}} + \tfrac{9}{5}\sigma_{\text{okt}}^2} - \sigma_y = 0 \tag{D1}$$

as given in equation (5) the radical can become imaginary. In the case of the Von Mises criterion, which according to Section A3 of Appendix A can be written as:

$$F = \sqrt{\tfrac{1}{6}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}} - \sigma_y = 0 \tag{D2}$$

this problem does not arise, since the expression under the root sign is always positive.

In equation (D1), on the other hand, in view of the term $\sigma_y\sigma_{\text{okt}}$ in which σ_{okt} can be negative and have a dominant effect, the whole expression under the root sign can become negative. From the physical point of view this does not constitute a problem, for we are within the yield surface, so that the material is still elastic and is in fact not concerned with yield surfaces at all. Practically, however, numerical problems are liable to arise. One way of avoiding these is to perform the calculations with the squared function as the yield surface.

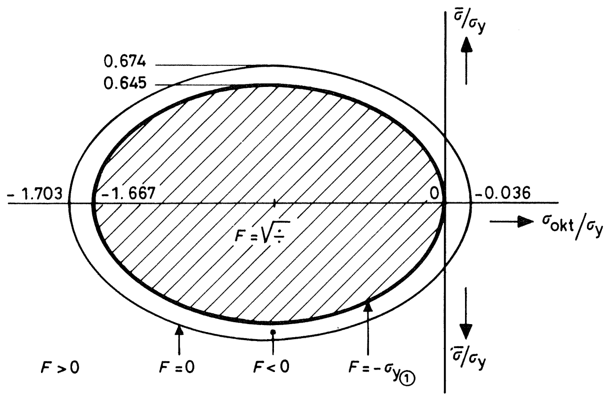


Fig. D1. The yield criterion causes a forbidden region.

Literature

1. BESSELING, J. F., A theory of elastic plastic and creep deformations of an initially isotropic material showing anisotropic strain-hardening, creep recovery, and secondary creep, Journ. Appl. Mech. Dec. (1958), pp. 529–536.
2. PRAGER, W., The theory of plasticity; a survey of recent achievements, Proc. Inst. Mech. Engrs. 169, (1955), pp. 41–57.
3. SHIELD, R. T. and H. ZIEGLER, On Prager's Hardening Rule, Zamp 9a, (1958), pp. 260–276.
4. ZIEGLER, H., A modification of Prager's hardening rule, Quart Appl. Math. 17, (1959), pp. 1-55.
5. NAYAK, G. C. and O. C. ZIENKIEWICZ, Elasto-plastic stress analysis a generalisation for various constitutive relations including strain softening, Int. Journ. Num. Meth. Eng. 5, (1972), pp. 113–135.
6. NAYAK, G. C. and O. C. ZIENKIEWICZ, Convenient form of stress invariants for plasticity, Proc. ASCE Journ. Struct. Div., Vol. 98, No. St 4, pp. 949–954, April 1972.
7. BUYUKOZTURK, O., Non linear analysis of reinforced concrete structures, Computers and structures, Vol. 7, pp. 149–156, (1976).
8. DE GROOT, A. K. and G. M. A. KUSTERS, Beschouwingen t.a.v. enige vloei- en scheurkriteria, TNO-IBBC rapport BI-78-19, (1978), (in Dutch).
9. DE GROOT, A. K. and G. M. A. KUSTERS, Constitutieve vergelijkingen, A26/78-03 TNO-IBBC rapport B-78-304, (1978), (in Dutch).
10. MEULENBELD, B. and W. K. BAART, Analyse 2, Uitgeverij H. Stam N.V., (in Dutch).