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Contents

A SIMPLIFIED APPROACH TO CREEP BUCKLING AND CREEP INSTABILITY OF STEEL-CONCRETE COLUMNS

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Notation

A_b	concrete cross-sectional area
e_m	theoretical thickness of concrete section
e_0	load eccentricity
E_1, E_2	stiffness parameters of rheological model
E_b	modulus of elasticity of concrete at age of 28 days
$E(\tau)$	modulus of elasticity of concrete at time $t=\tau$
F	load on column
F_c	short-term loadbearing capacity of column
F_k	short-term elastic buckling load
F_p	first-order collapse load
$F_{c\infty}$	long-term loadbearing capacity of column
F_{ks}	buckling load of column based on stiffness of the steel
$F_{k\infty}$	creep buckling load
$f(t)$	function
$g(t)$	function
$h(t)$	function
h	depth of concrete section
H	force on rheological model
k, k_1, k_2	stiffness parameters of rheological model
k_b, k_c, k_d	constituent factors of creep coefficient
k_e, k_t	
k_c	stiffness of concrete member
k_s	stiffness of steel member
L	length of column
M_c	moment in concrete section
M_s	moment in steel section
n	safety against elastic buckling
n_k	safety against creep buckling
N	frictional force of sliding element
N_c	normal force in concrete section
N_s	normal force in steel section
O	perimeter length of concrete section in contact with external atmosphere
r	ratio of stiffness of steel section to stiffness of whole section
t	point of time
α	auxiliary quantity
$\beta(t-\tau)$	creep strain as a function of load duration
$\beta_d(t-\tau)$	reversible creep strain as a function of load duration
$\beta_f(t-\tau)$	irreversible creep strain as a function of load duration
δ	deflection of column model
$\hat{\delta}$	initial deflection
Δ	compression of column model

ε	strain
η, η_1, η_2	viscosity parameters of rheological model
θ	parameter for creep rate
μ	auxiliary quantity
σ	stress
τ	point of time of load application
$\varphi, \varphi(\tau, t)$	creep coefficient as a function of t and τ
φ_d	coefficient for reversible creep strain (CEB-FIP 1978)
φ_f	coefficient for irreversible creep strain (CEB-FIP 1978)
φ_∞	creep coefficient for load of infinite duration
ψ	parameter for creep rate
ω	reinforcement percentage

A SIMPLIFIED APPROACH TO CREEP BUCKLING AND CREEP INSTABILITY OF STEEL-CONCRETE COLUMNS

Preface

This study was carried out at TNO-IBBC as part of the investigation program "steel-concrete columns", which is sponsered by CUR-VB.
The writer's wishes to express his gratitude to prof. ir. J. Witteveen, ir. A. C. W. M. Vrouwenvelder, ir. H. J. Buitenkamp and the members of the CUR-VB workinggroup C 15A for their stimulating suggestions and remarks.

Abstract

The effect of the creep of concrete on the physical and geometrical non-linear behaviour of concrete columns is dealt with in this article. The aim has been to achieve simplicity of treatment by choosing conveniently manipulatable mathematical models. The geometrical non-linear behaviour of columns is reduced to essentials with the aid of discrete kinematic models. The material strain behaviour of concrete is schematized by rheological models.

With the aid of these models the concepts of creep buckling and creep instability are analysed. By creep buckling is understood the academic case of a slowly buckling straight strut under axial (centric) load. Creep instability denotes the real phenomenon of sudden instability of a column occurring a long time after load application in consequence of its attaining a critical deflection due to time-dependent effects.

Further, the effect of the reinforcing steel in a concrete column upon the creep deformation behaviour of the column is investigated. The choice of model enables the problems involved to be properly visualized, and qualitative insight into the extent to which various parameters affect the creep deformation behaviour of concrete columns is obtained. The approach presented here can, for example, be useful in interpreting computer calculations or in assessing the creep sensitivity of a concrete structure.

A simplified approach to creep buckling and creep instability of steel-concrete columns

1 Introduction

The development of advanced numerical methods for the analysis of structures has made considerable progress in recent years. Various computer programs enable the present-day designer to analyse the behaviour of many kinds of structure in a realistic manner. There is, however, a risk that the overall picture will be obscured by a multitude of numerical data. Hence there continues to exist a need for simple analytical methods that can give deeper insight into the physical basis of structural behaviour. Particularly in the design stage it is important to know what phenomena will play a part in the structure to be designed and in what way they will do this.

Creep of concrete is a phenomenon of this kind with which it is sometimes very necessary to reckon in connection with the design of concrete structures. This applies to prestressed concrete or to concrete structures for which close dimensional control is required. Creep may also play a part in stability problems. For example, the stability of slender columns may be endangered as a result of time-dependent increase of an initial lateral deflection. This last-mentioned phenomenon will be dealt with in this article, in which it will be endeavoured to tackle the problems in their most elementary form with the aid of a simple model conception.

First, a short description of the creep behaviour of concrete will be given. Then the mathematical formulation of this behaviour with the aid of rheological models will be considered. The ostensive (directly demonstrative) character of such models makes them eminently suitable for the purpose of this article. Next, with the aid of a combination of rheological models and a kinematic model, an analysis of the concepts of "creep buckling" and "creep instability" is presented. Finally, the effect of reinforcing steel on the time-dependent deformation of a reinforced concrete column is studied.

2 Creep of concrete

2.1 *Linear creep*

When subjected to loading of long duration, concrete displays a delayed deformation which is called creep. The magnitude of this creep deformation may be a multiple of that of the deformation which occurs immediately on application of loading. For concrete subjected to constant compressive stress the time-dependent strain is approximately proportional to the stress. This is referred to as linear creep and is expressed by the following formula (see Fig. 1):

$$\varepsilon(t) = \frac{\sigma_0}{E(\tau)} + \varphi \frac{\sigma_0}{E_b} \qquad (1)$$

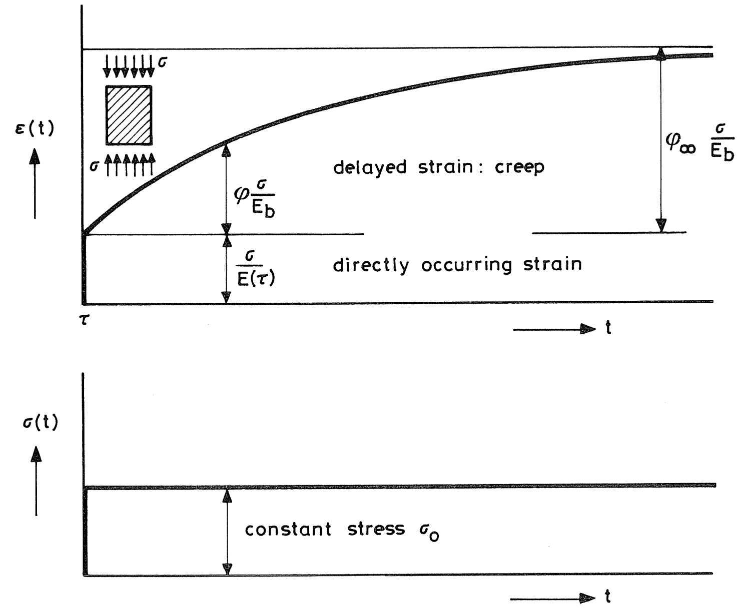


Fig. 1. Creep of concrete under constant stress.

where:

E_b = modulus of elasticity at a specified time, e.g., after 28 days' hardening
$E(\tau)$ = modulus of elasticity at the time of loading
$\varepsilon(t)$ = time-dependent strain
σ_0 = constant compressive stress
φ = creep coefficient

Formula (1) is valid only if the concrete stress is not more than about 50% of the compressive strength. At higher stresses the creep strains are relatively larger and there is then no longer a linear relation between stresses and (time-dependent) strains. In that case there is what is known as non-linear creep. This last-mentioned creep behaviour will not be considered here.

2.2 *Creep coefficient*

With the aid of formula (1) the (linear) creep of concrete can be characterized by a dimensionless creep coefficient φ. It is evident that this coefficient is primarily dependent on the duration of the load. Other factors that affect the magnitude of the creep coefficient are:

- the quality (strength class) of the concrete;
- the age of the concrete at the time of load application;
- the dimensions of the structural member;
- the moisture conditions within the concrete.

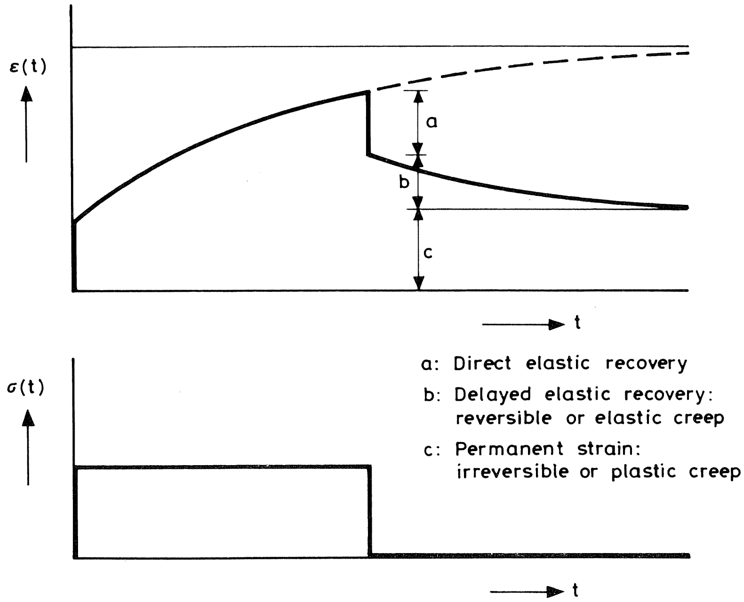


Fig. 2. Creep of concrete for successive loading and unloading.

If concrete which has been subjected to load for a time is then unloaded, part of the creep strain will in the long run be recovered, i.e., it will disappear, and part of the creep strain will remain permanently (Fig. 2).

After direct elastic recovery on removal of the load there occurs a further delayed recovery over a period of time. This part of the creep strain is called the elastic or reversible creep strain. The part that remains permanently after load removal is called the plastic or irreversible creep strain.

The creep of concrete can therefore be described as a combination of a delayed elastic and a delayed plastic strain behaviour.

The proportion of plastic creep strain in the total creep strain depends mainly on the age that the concrete has attained at the time of applying load to it. Very old concrete shows hardly any plastic creep strain. The magnitude of the elastic creep is more or less independent of the age of the concrete and can, in the light of the latest information, be taken as equal to about 40% of the directly occurring strain.

3 Mathematical formulation of creep

3.1 *General method*

The magnitude of φ can be determined experimentally as a function of the factors already mentioned. Quantitative data relating to φ are given in the various codes and directives for the design of concrete structures. The time-dependent deformations of a structure in which the concrete stresses remain constant can be determined quite simply with the aid of formula (1). But if the concrete stresses vary gradually or stepwise with time, there is a complication in that this formula can then not be directly applied.

In principle this problem can be solved as follows: Consider the varying concrete stress as if it were composed of an infinite number of constant loads, of magnitude $d\sigma$, applied in succession. Using the superposition principle and formula (1), we can thus write (see Fig. 3):

$$\varepsilon(t) = \sigma(0)\left\{\frac{1}{E(0)} + \frac{1}{E_b}\varphi(0,t)\right\} + \int_0^t \frac{d\sigma(\tau)}{dt}\left\{\frac{1}{E(\tau)} + \frac{1}{E_b}\varphi(\tau,t)\right\} d\tau \tag{2}$$

In formula (2) the creep coefficient φ is conceived as a function of two variables, namely, the load duration t and the point of time τ at which the load is applied. Hence: $\varphi = \varphi(\tau, t)$. For a given stress behaviour $\sigma(t)$ as a function of time the time-dependent

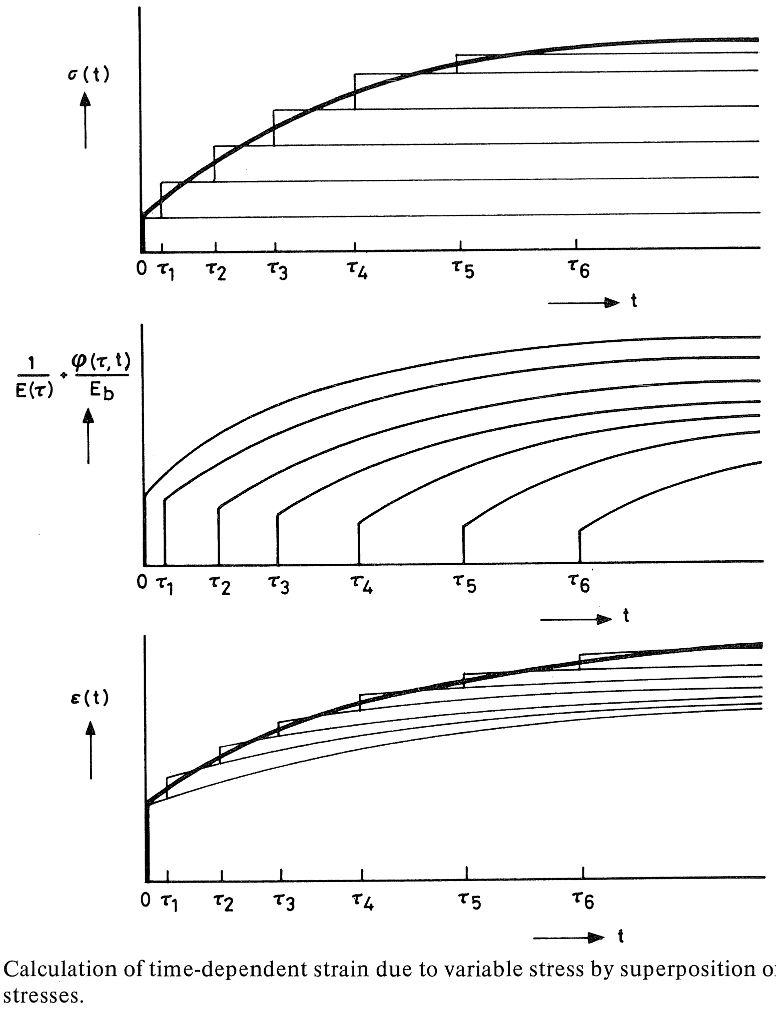


Fig. 3. Calculation of time-dependent strain due to variable stress by superposition of constant stresses.

strain $\varepsilon(t)$ is obtained by direct integration of (2). In cases where the strain is given and the stress is to be calculated – e.g., in the case of relaxation – an integral equation will have to be solved. This occurs also in dealing with geometrical non-linear problems in which σ and ε are, in the first instance, unknown but which preserve a certain interrelationship. In general, it is not possible to obtain simple analytical solutions by this method.

Instead, it soon becomes necessary to have recourse to numerical methods. Only by a special choice of $\varphi(\tau, t)$ and $E(\tau)$ can the mathematical problems be reduced to manageable proportions. Of course, such a choice must agree as closely as possible with the actual – experimentally determined – creep behaviour of concrete.

In the following treatment the mathematical formulation of the creep of concrete will be tackled in a different way. Afterwards it will emerge that the procedure to be adopted here is essentially a question of making a suitable choice for $\varphi(\tau, t)$ and $E(\tau)$. But the way in which this is done provides a better perception of the physical events than is provided by the formally mathematical approach in accordance with formula (2).

3.2 *Rheological models*

The creep behaviour of concrete can be made amenable to analytical study by means of so-called rheological models. These are mechanical models which have, inter alia, successfully been applied to the schematization of the visco-elastic behaviour of plastics and of metals at high temperatures (see Fig. 4). The basic elements of these models are the spring and the dashpot.

The relation between stress and strain for a linearly elastic material (conforming to Hooke's law) can be schematized by means of single spring. It is expressed by the formula:

$$\sigma = E\varepsilon \qquad (3)$$

A piston pierced by small openings and moving in a cylinder filled with a fluid (e.g., oil) forms a damping device called a dashpot which can suitably schematize the relation between stresses and strains of a purely viscous material (Newtonian substance).

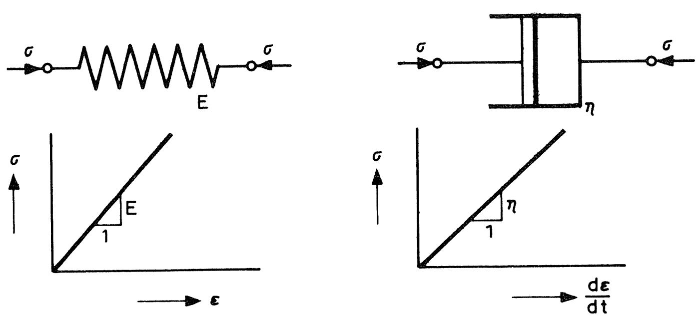


Fig. 4. The basic elements of rheological models: spring and dashpot.

Application of a load to this model does not produce a directly occurring strain, but it does produce a rate of strain. Hence there is a time-dependent strain behaviour expressed by:

$$\sigma = \eta \frac{d\varepsilon}{dt} \quad (4)$$

The two elementary rheological models that can be built up from these basic elements are the Maxwell model and the Kelvin-Voight model respectively. These comprise a spring and a dashpot, connected in series and in parallel respectively (see Fig. 5).

The strain behaviour of these models can be characterized by the strain-time diagram for loading and then unloading. Under sustained load the Maxwell model can in principle attain an infinitely large strain and will on unloading display only a directly occurring elastic recovery. The time-dependent strain behaviour of this model can be described as delayed plastic. The Kelvin-Voight model, on the other hand, is characterized by a delayed elastic strain behaviour. A sustained constant load will in the end result in a finite strain of this model. On unloading, this strain will in the end be completely recovered, i.e., it will disappear.

3.3 *Maxwell model with time-dependent parameters; irreversible creep*

Both the rheological models shown in Fig. 5 are too incomplete for schematizing the material behaviour of concrete. This situation can be improved by conceiving the properties of the spring and the dashpot as time-dependent. The implications of this will be elaborated for the Maxwell model.

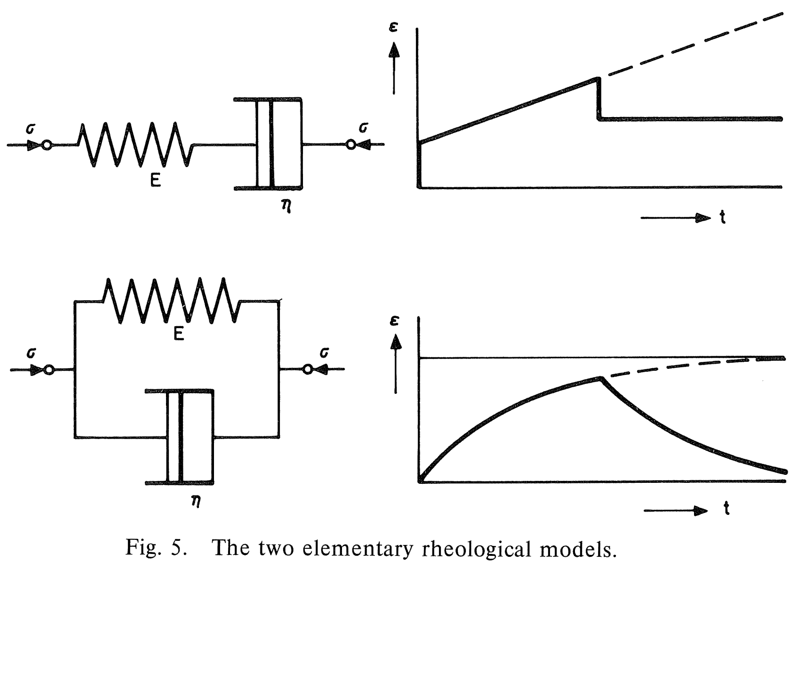


Fig. 5. The two elementary rheological models.

The constitutive equation of a spring with a time-dependent stiffness is: *

$$\frac{d\sigma}{dt} = E(t)\,\frac{d\varepsilon}{dt} \tag{5}$$

while for dashpot:

$$\sigma = \eta(t)\,\frac{d\varepsilon}{dt} \tag{6}$$

The series connection of spring and dashpot is characterized by the constitutive equation:

$$\frac{d\varepsilon}{dt} = \frac{1}{E(t)}\,\frac{d\sigma}{dt} + \frac{1}{\eta(t)}\,\sigma \tag{7}$$

With the aid of (7) it is possible to make a comparison between the time-dependent strain behaviour of this model and that of concrete as defined in Chapter 2. To this end, equation (7) is solved for a constant load $\sigma = 1$ applied at time $t = \tau$:

$$\varepsilon(\tau) = \frac{1}{E(\tau)}$$

$$\frac{d\varepsilon}{dt} = \frac{1}{\eta(t)}$$

On integrating, we obtain:

$$\varepsilon(t) = \frac{1}{E(\tau)} + f(t) - f(\tau) \tag{8}$$

where:

$$f(t) = \int_0^t \frac{1}{\eta(\xi)}\,d\xi$$

The behaviour of $\varepsilon(t)$ for several values of τ is indicated in Fig. 6. From equation (8) it follows directly that all the curves in the diagram are parallel to one another. The choice of the shape of these curves is free, subject to their having an upper limit formed by a horizontal asymptote, since the creep strain will in the end attain a certain finite magnitude. It then further follows from (8) that, for large values of τ, the value of $f(t) - f(\tau)$ will, at the limit, approach zero. According as the age of this model increases, it will finally cease to exhibit creep behaviour. This last-mentioned inference is not in agreement with the actual observed creep behaviour of concrete. Even very old concrete does still exhibit some - though modest - creep behaviour.

The curves in Fig. 6 are sometimes called creep functions. A creep function describes the time-dependent strain behaviour of a concrete fibre which is subjected to a unit load

* $\sigma = E(t)\varepsilon$ is incorrect. An increase of the modulus of elasticity E with the passage of time has no effect on strains due to loads applied earlier on.

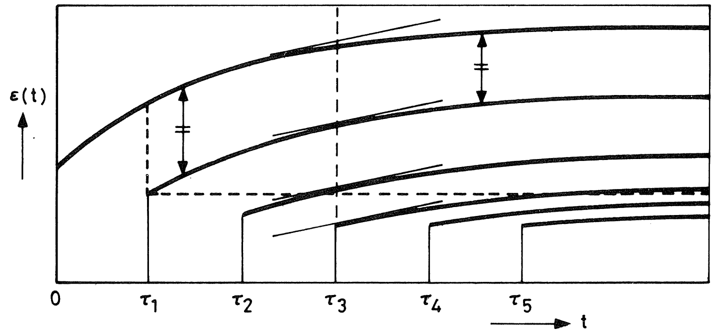


Fig. 6. Parallel creep function curves for the Maxwell model with time-dependent parameters. The dotted line indicates the time-dependent strain behaviour which occurs if, after load application at $t=0$, load removal takes place at $t=\tau_1$.

applied at certain point of time. In practical calculations the assumption of parallel creep functions is often made as a starting point. In the literature this method of analysis is known as the "rate of creep method". Its mathematical basis is relatively simple.

Comparison of formulas (1) and (8) shows that:

$$f(t)=\frac{\varphi(t)}{E_b}$$

Furthermore, using the expression:

$$f(t)=\int_0^t \frac{1}{\eta(\xi)}\,\mathrm{d}\xi$$

it follows that:

$$\frac{1}{\eta(t)}=\frac{1}{E_b}\frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

and substitution of this into (7) finally gives:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=\frac{1}{E(t)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}+\frac{1}{E_b}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\sigma \tag{9}$$

The constitutive equation for concrete has thus been reduced to a simple differential equation instead of the awkward integral equation (2).

Equation (9), known as Dischinger's equation, was derived with the aid of a rheological model. The reader can verify for himself that this equation can alternatively be obtained from equation (2) by substitution into it of: $\varphi(\tau,t)=\varphi(t)-\varphi(\tau)$ and then differentiating the equation with respect to t.

3.4 *Rheological model with three parameters; reversible creep*

The Maxwell model has an entirely irreversible creep behaviour. Creep strain, once it has been attained, remains permanently after load removal. On the other hand, an entirely reversible creep behaviour is obtained with a three-parameter model as shown in Fig. 7. It comprises a single spring connected in series with a Kelvin unit. The parameters of the Kelvin unit are time-independent, so that the creep behaviour of this model is independent of the "age" of the model at the time of loading.

The constitutive equation of the three-parameter model in Fig. 7 is as follows:

$$\frac{d^2\varepsilon}{dt^2}+\frac{E_2}{\eta}\frac{d\varepsilon}{dt}=\frac{1}{E_1}\frac{d^2\sigma}{dt^2}+\left(\frac{1}{\eta}+\frac{E_2}{\eta E_1}+\frac{dE_1^{-1}}{dt}\right)\frac{d\sigma}{dt} \qquad (10)$$

and with the aid of this equation it can be shown that the family of creep functions associated with this model is represented by:

$$\varepsilon(t)=\frac{1}{E_1(\tau)}+\frac{1}{E_2}\left(1-e^{-\frac{E_2}{\eta}(t-\tau)}\right) \qquad (t>\tau) \qquad (11)$$

These curves have been plotted in Fig. 8. From equation (11) it follows directly that the creep strain of the model depends only on the load duration $t-\tau$. The finally attained creep strain is determined by the magnitude of E_2 and the creep rate by the quotient E_2/η.

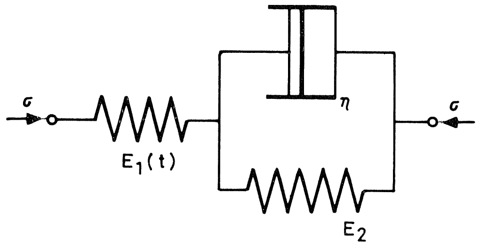


Fig. 7. Rheological model with three parameters; reversible creep behaviour.

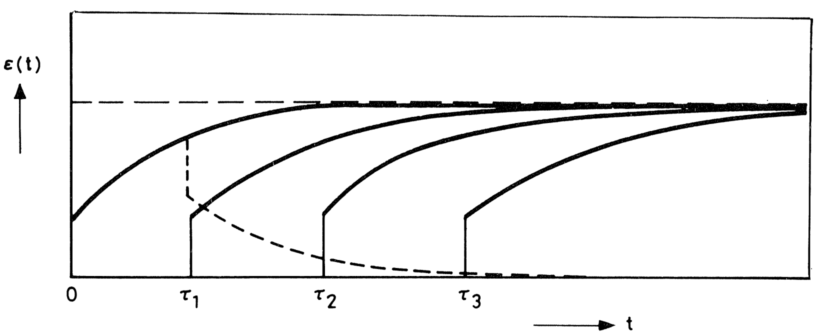


Fig. 8. Creep functions associated with the three-parameter model. For the sake of clarity, in this diagram, the magnitude of the directly occurring strain has been taken as independent of the point time τ at which load is applied. The behaviour of the time-dependent strain is indicated by the dotted line, for load application at $t=0$ and load removal at $t=\tau_1$.

With the aid of the substitutions:

$$\theta = \frac{E_2}{\eta} \tag{12a}$$

$$\varphi_\infty = \frac{E_b}{E_2} \tag{12b}$$

$$E(t) = E_1(t) \tag{12c}$$

equation (11) can be written as:

$$\varepsilon(t) = \frac{1}{E(\tau)} + \frac{1}{E_b}\varphi_\infty (1 - e^{-\theta(t-\tau)}) \tag{12d}$$

The creep coefficient $\varphi(\tau, t)$ of this model is therefore given by:

$$\varphi(\tau, t) = \varphi_\infty \beta(t - \tau) \tag{13}$$

where:

$$\beta(t-\tau) = 1 - e^{-\theta(t-\tau)}$$

while φ_∞ is the creep coefficient for load sustained for an infinitely long period and θ is a measure for the rate of strain. These are quantities that can be measured on a concrete specimen. Hence, with the aid of equations (12a) to (12c), the model can be "calibrated" with regard to the experimentally determined creep behaviour of concrete.

Expressed in φ_∞ and θ, the constitutive equation (10) becomes:

$$\frac{d^2\varepsilon}{dt^2} + \theta\frac{d\varepsilon}{dt} = \frac{1}{E(t)}\frac{d^2\sigma}{dt^2} + \left(\frac{\theta\varphi_\infty}{E_b} + \frac{\theta}{E(t)} + \frac{dE_{(t)}^{-1}}{dt}\right)\frac{d\sigma}{dt} \tag{14}$$

If the strain rate of the model is zero, the dashpot can be omitted, i.e., the force in the dashpot is then likewise zero. The creep strain at that instant is determined by the connection in series of two springs. For calculating the delayed elastic strain in such a case an acceptable simple approach is to base oneself on the stiffness E_1 for load of short duration and to reduce this stiffness by applying a factor $1 + \varphi_\infty$. This simple method is known as the "effective modulus method".

3.5 *Rheological model with four parameters; CEB-FIP creep function*

The models considered in the two preceding sections of this article have either an entirely irreversible or an entirely reversible creep behaviour.

As has been noted in Chapter 2, the creep of concrete comprises a reversible and an irreversible part, however. An obvious approach in seeking to schematize this behaviour consists in combining the two above-mentioned models. The four-parameter model obtained in this way is shown in Fig. 9. It comprises a Maxwell unit with time-dependent parameters connected in series with a Kelvin unit with time-independent

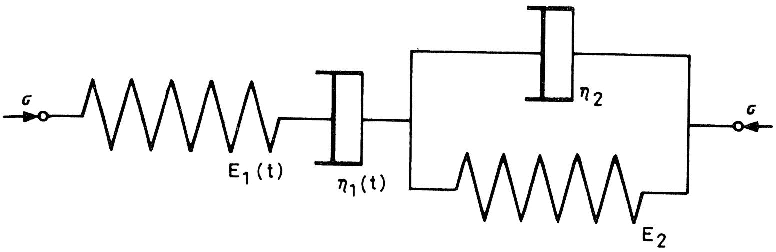


Fig. 9. Rheological model with four parameters; combination of reversible and irreversible creep behaviour.

parameters. The Maxwell unit is responsible for the irreversible, the Kelvin unit for the reversible part of the creep.

The "age" of the model at the time of loading affects the magnitude of the irreversible creep, but not that of the reversible creep. The "CEB-FIP Model Code for Concrete Structures", 1978 [1], proposes this rheological model for schematizing the creep behaviour of concrete.

The constitutive equation of the four-parameter model in Fig. 9 is as follows:

$$\frac{d^2\varepsilon}{dt^2}+\frac{E_2}{\eta_2}\frac{d\varepsilon}{dt}=\frac{1}{E_1(t)}\frac{d^2\sigma}{dt^2}+\left(\frac{1}{\eta_1(t)}+\frac{1}{\eta_2}+\frac{E_2}{\eta_2 E_1(t)}+\frac{dE_1^{-1}(t)}{dt}\right)\frac{d\sigma}{dt}+\left(\frac{E_2}{\eta_2 n_1(t)}+\frac{d\eta_1^{-1}(t)}{dT}\right)\sigma \tag{15}$$

and the family of creep functions expressed in the parameters $E_1(t)$, E_2, $\eta_1(t)$, η_2 is represented by:

$$\varepsilon(t)=\frac{1}{E_1(\tau)}+\int_\tau^t\frac{1}{\eta_1(\xi)}\,d\xi+\frac{1}{E_2}\left(1-e^{-\frac{E_2}{\eta_2}(t-\tau)}\right) \tag{16}$$

The "CEB-FIP" creep function is of the following form:

$$\varepsilon(t)=\frac{1}{E(\tau)}+\frac{\varphi_f}{E_b}\{\beta_f(t)-\beta_f(\tau)\}+\frac{\varphi_d}{E_b}\beta_d(t-\tau) \tag{17}$$

where:

φ_d = coefficient for reversible creep strain = 0.4

φ_f = coefficient for irreversible creep strain, depending on relative humidity and cross-sectional dimensions (see Fig. 10)

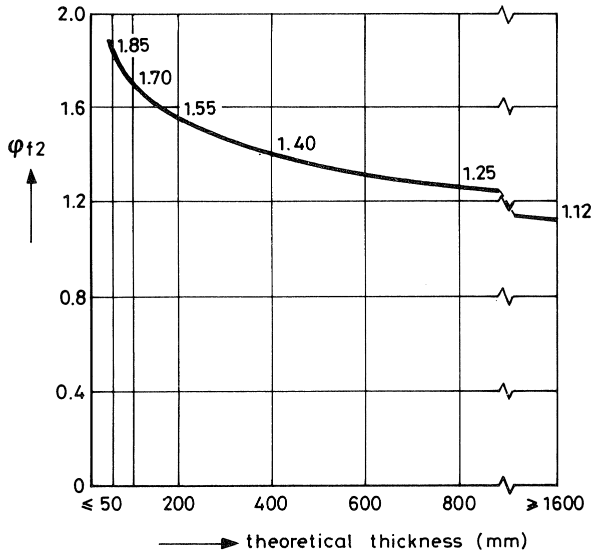
β_d = a function which describes the reversible creep strain behaviour in relation to time; in ref. [1] this function is given in graph form; an earlier CEB-FIP report [2] gives a formula:

$\beta_d = 1 - e^{-\theta(t-\tau)}$ where $\theta = 0.02$ [days^{-1}]

β_f = a function which describes the irreversible creep strain behaviour in relation to time; in ref. [2] this function is given as:

ambient environment 1	relative humidity 2	coefficients of creep φ_{f1} 3	coefficients of shrinkage ε_{s1} 4	coefficient λ 5
water		0.8	+0.00010	30
very damp atmosphere	90%	1.0	−0.00013	5
outside in general	70%	2.0	−0.00032	1.5
very dry atmosphere	40%	3.0	−0.00052	1

$\varphi_f = \varphi_{f1} \cdot \varphi_{f2}$; φ_{f1} from (a), φ_{f2} from (b)



theoretical thickness* $\lambda(2A/0)$ (mm)	ψ (days)
⩽ 50	300
100	400
200	600
400	1000
800	1600
⩾ 1600	2800

* A = cross-sectional area
0 = perimeter
λ = from (a)

Fig. 10. Calculation of creep coefficient according to "CEB-FIP Model Code for concrete structures", April 1978.
From (a) and (b) is obtained the magnitude of the coefficient for irreversible creep for load of infinite duration. From (c) is obtained the magnitude of the coefficient ψ in the function β_f in formula (17); ψ is a measure of the rate of irreversible creep strain.

$$\beta_f(t) = \left(\frac{t}{t+\psi}\right)^{1/3}$$

where ψ depends on the cross-sectional dimensions (see Fig. 10c)

By equating (16) and (17) the parameters $E_1(t)$, E_2, $n_1(t)$ and n_2 of the rheological model in Fig. 9 can be expressed in the above quantities:

$$E_1(t) = E(t) \qquad \eta_1(t) = E_b\left(\varphi_f \frac{d\beta_f}{dt}\right)^{-1}$$

$$E_2(t) = E_b(\varphi_d)^{-1} \qquad \eta_2 = E_b(\theta\varphi_d)^{-1}$$

Thus the four-parameter model has been calibrated for the creep formulation for concrete as given in the "CEB-FIP Model Code". On substitution of the calibrated values of the four parameters into equation (15) the constitutive equation is obtained in the following form:

$$\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2}+\theta\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=\frac{1}{E(t)}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}+\left(\frac{\varphi_f}{E_b}\frac{\mathrm{d}\beta_f}{\mathrm{d}t}+\frac{\theta\varphi_d}{E_b}+\frac{\theta}{E(t)}+\frac{\mathrm{d}E_{(t)}^{-1}}{\mathrm{d}t}\right)\frac{\mathrm{d}\sigma}{\mathrm{d}t}+\left(\frac{\theta\varphi_f}{E_b}\frac{\mathrm{d}\beta_f}{\mathrm{d}t}+\frac{\varphi_f}{E_b}\frac{\mathrm{d}^2\beta_f}{\mathrm{d}t^2}\right)\sigma \tag{18}$$

3.6 *Rheological model with three time-dependent parameters*

A combination of reversible and irreversible creep behaviour can also be schematized by means of a three-parameter model with time-dependent parameters, as shown in Fig. 11. This can be done by letting the stiffness of the spring in the Kelvin unit increase up to a constant value as a function of time. By also letting the viscosity of the dashpot increase as a function of time, a realistic creep model for concrete is obtained. On loading and then, after some time, unloading of the model, the creep strain that has occurred will in part be permanent, since the spring in the Kelvin unit has become stiffer during the load period and undergoes less deformation on being relieved of load than on being loaded. With increasing "age" of the model the creep behaviour changes from combined reversible and irreversible to completely reversible. This is more particularly because the parameters of the spring and dashpot in the Kelvin unit become time-independent in the long run, with the result that the model will, for large values of τ, turn into the three-parameter model of Section 3.4. These behaviour patterns will now be dealt with in more detail.

The constitutive equation of the three-parameter model is:

$$\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2}+\left(\frac{E_2}{\eta}+\frac{1}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=\frac{1}{E_1}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}+\left(\frac{1}{\eta}+\frac{E_2}{\eta E_1}+\frac{1}{\eta E_1}\frac{\mathrm{d}\eta}{\mathrm{d}t}+\frac{\mathrm{d}E_1^{-1}}{\mathrm{d}t}\right)\frac{\mathrm{d}\sigma}{\mathrm{d}t} \tag{19}$$

and the family of creep functions associated with this model is described by:

$$\varepsilon(t)=\frac{1}{E_1(\tau)}+g(\tau)\{h(t)-h(\tau)\} \tag{20}$$

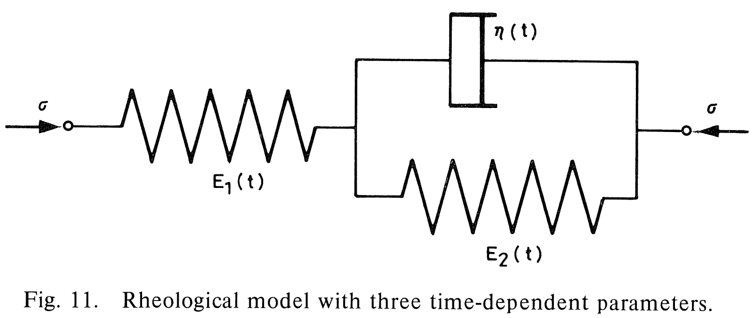


Fig. 11. Rheological model with three time-dependent parameters.

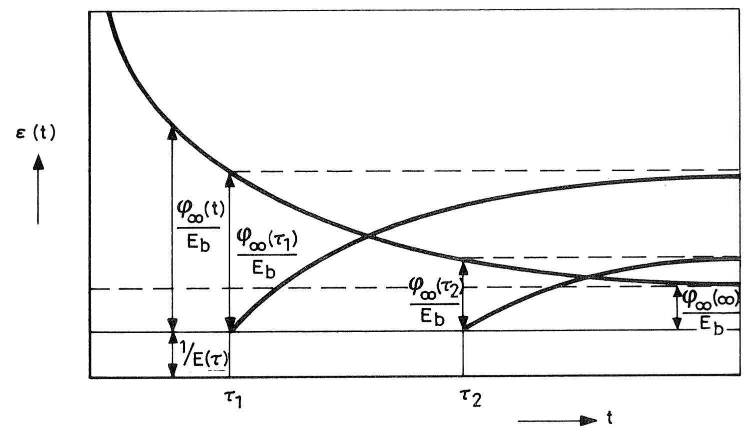


Fig. 12. Creep functions of a rheological model with three time-dependent parameters. For the sake of clarity, in this diagram, the magnitude of the directly occurring strain has been taken as independent of the time τ at which load is applied.

where:

$$g(\tau) = \exp \int_0^{\tau} \frac{E_2(\xi)}{\eta(\xi)} \, d\xi \quad \text{and} \quad h(t) = \int_0^{t} \left\{ \frac{1}{\eta(\xi)} (g(\xi))^{-1} \right\} d\xi$$

A particular choice for $E_1(t)$, $E_2(t)$ and $\eta(t)$ enables (20) to be written in the following form:

$$\varepsilon(t) = \frac{1}{E(\tau)} + \frac{1}{E_b} \varphi_{\infty}(\tau) \beta(t-\tau) \tag{21}$$

where $\varphi_{\infty}(\tau)$ is an arbitrary function and $\beta(t-\tau)$ is given by:

$$\beta(t-\tau) = 1 - e^{-\theta(t-\tau)} \tag{21a}$$

The family of creep functions expressed by (21) has been plotted in Fig. 12.

$\varphi_{\infty}(\tau)$ is the creep coefficient for load sustained for an infinitely long time. With the three-parameter model shown in Fig. 11 it is possible to choose this coefficient arbitrarily, depending on the time of load application τ. The effect of the age of the concrete on the creep behaviour can therefore be incorporated into this coefficient $\varphi_{\infty}(\tau)$, which, on the evidence of the experimentally determined creep behaviour of concrete, has to be a decreasing function of τ which becomes constant for large values of τ. In this way two important creep properties of concrete are taken into account, namely, the lower creep values with increasing "age" and the reversible creep behaviour of very old concrete. The curve representing $\varphi_{\infty}(\tau)$ has likewise been plotted in Fig. 12.

The functions $E_2(t)$ and $\eta(t)$ can be expressed in $\varphi_{\infty}(t)$ and θ. On equating (20) and (21) the following are obtained:

$$\eta(t) = \frac{E_b}{\theta \varphi_{\infty}(t)} \tag{22}$$

$$E_2(t) = \frac{E_b}{\theta\varphi_\infty(t)}\left(\theta + \frac{1}{\varphi_\infty(t)}\frac{\mathrm{d}\varphi_\infty(t)}{\mathrm{d}t}\right) \tag{23}$$

On examination of (22) and (23) in more detail it is found that $\eta(t)$ and $E_2(t)$ do indeed increase with the passage of time, finally approaching constant values. The creep behaviour of the model for large values of t corresponds to that of the three-parameter model considered in Section 3.4.

The creep function (21) typifies the rheological model discussed here. A characteristic feature is that the creep coefficient $\varphi(\tau, t)$ in this case is composed of a product of two factors φ_∞ and β which are a function of τ and of $t-\tau$ respectively. In "Principles and Recommendations CEB-FIP, June 1970" [3] the creep coefficient $\varphi(\tau, t)$ is also given as a product of factors:

$$\varphi(\tau, t) = k_c \cdot k_b \cdot k_e \cdot k_d \cdot k_t \tag{24}$$

where:

k_c = a coefficient depending on the relative humidity of the atmosphere
k_b = a coefficient depending on the quality of the concrete
k_e = a coefficient depending on the theoretical thickness of the section
k_d = a coefficient depending on age of the concrete at the time of loading
k_t = a coefficient depending on the load duration

Numerical data relating to these factors are presented in graph form in ref. [3] (see Fig. 13). In the Netherlands code of practice for concrete [4] the method given for calculating the creep coefficient $\varphi(\tau, t)$ is virtually the same.

That code moreover gives a formula for determining k_t:

$$k_t = \frac{1000(t-\tau)}{1000(t-\tau) + 40\sqrt{e_m^3}} \tag{25}$$

where:

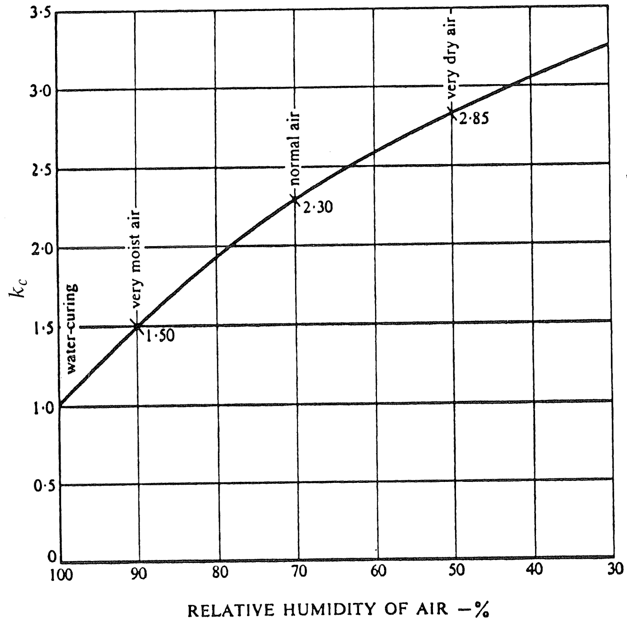
$e_m = 2A_b/O$

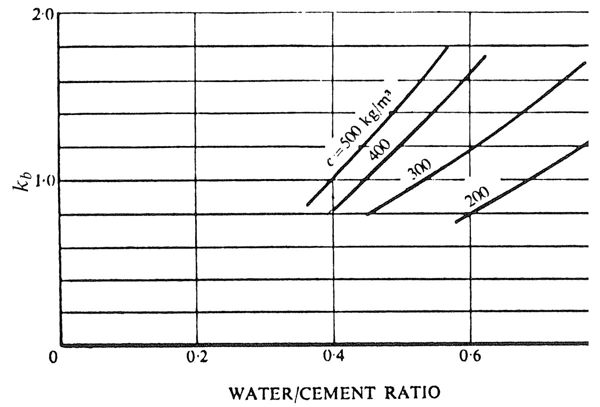
and:

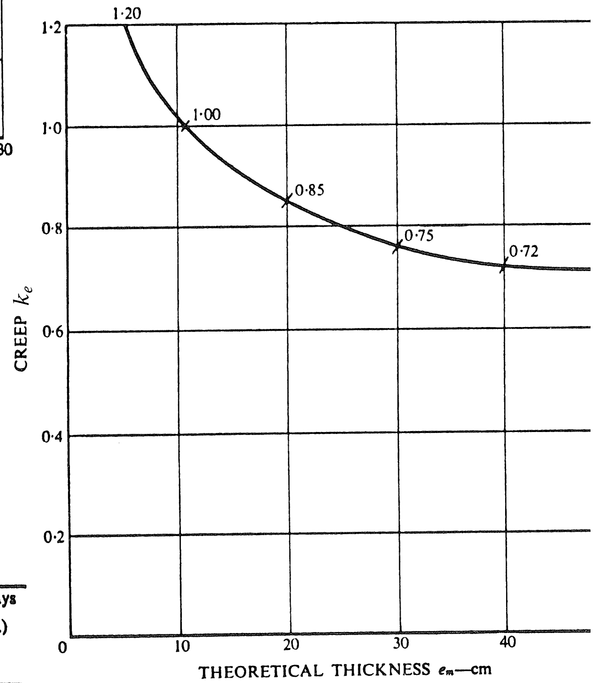
$t-\tau$ = the load duration in days
e_m = the theoretical thickness of the concrete section, in mm
A_b = the cross-sectional area of the concrete, in mm^2
O = that part of the perimeter of the concrete section which is in contact with the atmosphere, in mm

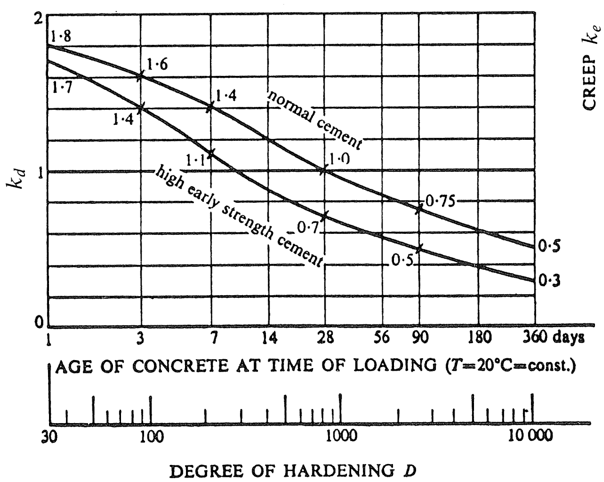
It is evident that this creep formulation proceeds from a schematization of the creep behaviour of concrete by means of a rheological model with three time-dependent parameters. The product of the factors k_c, k_b, k_e and k_d in (24) corresponds to the function $\varphi_\infty(\tau)$ in equation (21):

$$\varphi_\infty(\tau) = k_c \cdot k_b \cdot k_e \cdot k_d \tag{26}$$









$$\varphi(\tau,t) = k_c \cdot k_b \cdot k_e \cdot k_d \cdot k_t$$

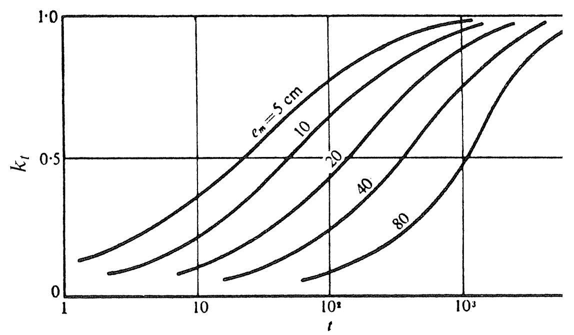


Fig. 13.
Calculation of creep coefficient according to "CEB-FIP Principles and Recommendations", June 1970.

while the factor k_t in equation (24) corresponds to the function $\beta(t-\tau)$ in equation (21). The function $k_t(t-\tau)$ in equation (25) does not, however, posses the same mathematical form as $\beta(t-\tau)$ in equation (21a), but they do both have a horizontal asymptote $k_t(\infty)=1$ and $\beta(\infty)=1$ respectively. The agreement between the three-parameter model in this Section and the creep formulation in ref. [4] is therefore not quite exact. Equations (21a) and (25) can, however, be made to agree with one another as closely as possible by letting the curves representing them touch each other at the origin.

In this way a relation between the parameter θ of the rheological model and the theoretical thickness of a concrete section is obtained:

$$\theta=\frac{25}{\sqrt{e_m^3}} \tag{27}$$

With equations (26) and (27) the relation has been established between the three-parameter model envisaged in the present Section and the creep formulation in refs. [3] and [4].

The constitutive equation (19), rewritten in the parameters $E(t)$, $\varphi_\infty(t)$ and θ is obtained by substitution of (22) and (23) into (19):

$$\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2}+\theta\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=\frac{1}{E(t)}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}+\left(\frac{\theta\varphi_\infty(t)}{E_b}+\frac{\theta}{E(t)}+\frac{\mathrm{d}E_{(t)}^{-1}}{\mathrm{d}t}\right)\frac{\mathrm{d}\sigma}{\mathrm{d}t} \tag{28}$$

3.7 *Concluding remarks*

The use of rheological models for schematizing the creep of concrete is not new. On the contrary, already in a very early stage of modern research on creep the usefulness of such models was shown by various investigators (for example, see ref. [5]).

All the same, in actual practice, rheological models are not well known, more particularly in relation to national and international codes of practice for concrete design and construction. The present author accordingly considered it relevant to deal with these matters rather more fully in these first few chapters than is necessary for the further treatment of the subject of this article.

Incidentally, it is to be noted that not too literal a physical interpretation should be given to rheological models. For example, the hardened cement paste between the gravel particles could conceivably be mathematically represented by dashpots and the gravel skeleton by springs. Such an approach to the problem is now obsolete, however. Material technologists have established advanced approximations of the creep behaviour of concrete, relying on a definite physical basis.

The object of using rheological models is to give visualizable form to a mathematical formulation. More particularly in connection with the analysis of the structural consequences of creep an approach of this kind can be helpful by clarifying the problem.

4 Creep buckling, creep instability

4.1 *Introduction*

In the preceding chapter some models where discussed with which the strain behaviour of concrete can be schematized. The present chapter is concerned with the consequences of this strain behaviour to the deformation behaviour of the concrete structure as a whole, with special reference to those parts thereof which are loaded in compression.

As is well known, the behaviour of compression members in a structural system is governed by two phenomena, namely, geometrical non-linearity and physical non-linearity. The pattern of force distribution in the structure is affected by the deformations (second-order effects) and the cross-sectional deformations are not directly proportional to the forces acting at the cross-sections concerned. Both phenomena cause the relation between load and deformation of such a member to deviate from linearity. In the next Section of this chapter a simple model analysis will be presented to clarify these effects. Next, in Sections 4.3 and 4.4 the influence of creep upon the geometrical and physical non-linear structural behaviour will be examined.

4.2 *Buckling, instability*

The geometrical and physical non-linear behaviour of structures can be analysed in a clear and readily visualizable manner with the aid of simple discrete kinematic models [6].

The simplest case is that of the concrete column fixed at the base (see Fig. 14). The kinematic models illustrated in Figs. 14b and 14c can be used for analysing the short-term deflection behaviour of the column.

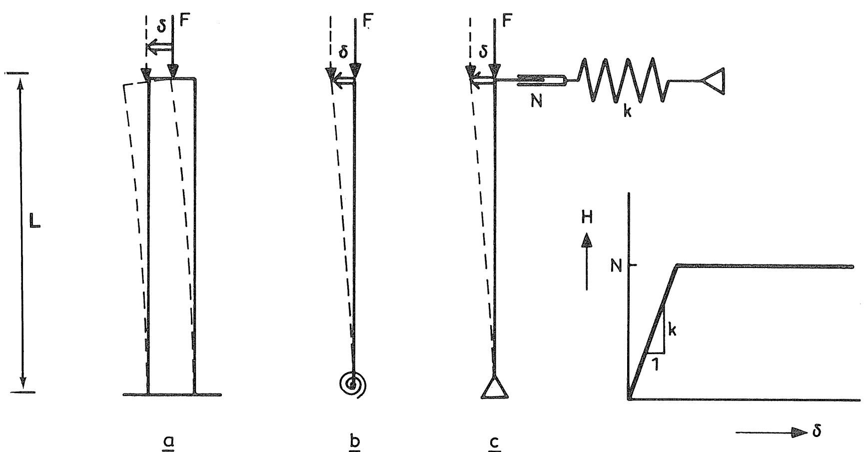


Fig. 14. Analysis of time-independent non-linear effects with the aid of simple kinematic models.

In these models the column is schematized to an infinitely rigid strut with its deformation properties concentrated at one point. In Fig. 14b this point is at the base; in Fig. 14c the deformation properties are schematized by a deformation element located at the top of the column. In order to take account of time-independent physical non-linearity, a certain deformation characteristic can be assigned to the deformation elements in the models, e.g., a bilinear relation between forces and deformations.

In the model in Fig. 14c this is achieved by building up the deformation element from a linear-elastic spring (with stiffness k) and a sliding element (with frictional force N). So long as the force in the deformation element is smaller than N the deformation behaviour is determined by the spring. But as soon as the force is equal to, or larger than, N the frictional force in the sliding element is exceeded and the deformation can then increase indefinitely. With the models in Figs. 14b and 14c it is possible, in a simple manner, to analyse the non-linear effects in structures composed of elasto-plastic materials. This will be further elaborated for the model in Fig. 14c.

Consider the model in the condition where the strut is given a deflection δ from the upright position. The horizontal force in the deformation element is then equal to $k\delta$, and if the model is in equilibrium in that condition, the following must apply:

$$F \cdot \delta = k \cdot \delta \cdot L$$

Hence the buckling load of the model is:

$$F_k = k \cdot L \tag{29}$$

while the deflection δ is indefinite.

If the strut already has an initial deflection $\hat{\delta}$, no buckling will occur when the vertical load is increased, but a non-linear relation between F and δ is found.

From considerations of equilibrium it follows that:

$$\delta = \frac{n}{n-1}\,\hat{\delta} \quad \text{where} \quad n = F_k/F \qquad (F_k = kL) \tag{30}$$

In this expression the factor of safety against elastic buckling of the strut is n. F increases the initially existing deflection by a factor $n/n-1$.

Since the magnitude of the horizontal force in the deformation element is limited by the frictional force N of the sliding element, the strut can, at a certain load F, undergo only a limited deflection δ_{max} because otherwise instability will occur.

From considerations of equilibrium we obtain:

$$\delta_{max} = \frac{NL}{F} \tag{31}$$

The relations (29), (30) and (31) are represented graphically in a load-deflection diagram in Fig. 15. The largest possible load F_c with which the model is still able to establish equilibrium is obtained by equating the expressions (30) and (31):

$$\frac{1}{F_c} = \frac{1}{F_k} + \frac{1}{F_p} \tag{32}$$

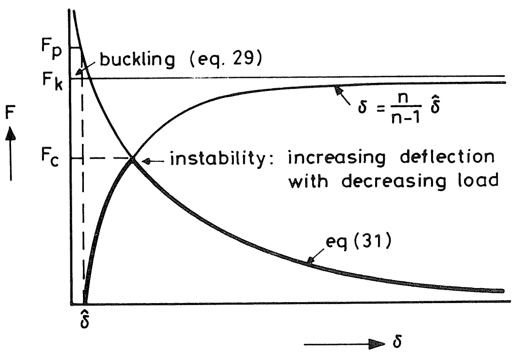


Fig. 15. Load-deflection diagram.

where F_p is the so-called first-order collapse load, as expressed by:

$$F_p = \frac{NL}{\hat{\delta}} \tag{33}$$

Formula (32) is known as Merchant's formula in the literature.

The designation "first-order collapse load" derives from collapse load analysis. More particularly, it is the collapse load which is found in a first-order analysis, i.e., an analysis which does not take account of geometrical non-linearity. If the spring is omitted from the kinematic model in Fig. 14c, the collapse load of the resulting model is expressed by (33).

The sensitivity of a structural member with regard to second-order effects can be estimated quite simply with the aid of equation (32). If F_k and F_p are of the same order of magnitude, second-order effects may play a part. On the other hand, for example, if F_k is large in relation to F_p, second-order effects need not be considered in calculating the loadbearing capacity.

4.3 *Creep buckling and creep instability with reversible creep*

The results obtained in the preceding Section are applicable to the short-term deformation behaviour of the column in Fig. 14c. In order to analyse the effect of delayed deformations of concrete upon the deflection behaviour of the column, the model shown in Fig. 16 can be used, if the concrete is in the uncracked state.

In this example a rheological model with three time-independent parameters has been incorporated in the deformation element in which the deformation behaviour of the column is assumed to be concentrated.

The load-deflection diagram of the column model is shown in Fig. 17. The point of time at which the condition of the model is considered is of importance. For load of short duration ($t = \tau$), equation (30) holds, with $k = k_1$.

For load of long duration ($t = \infty$) the magnitude of the deflection can be calculated from:

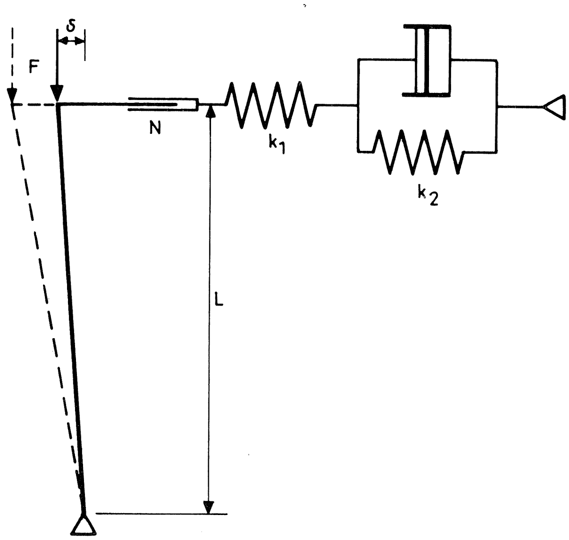


Fig. 16. Kinematic model for studying the effect of reversible creep on the geometrical non-linear behaviour.

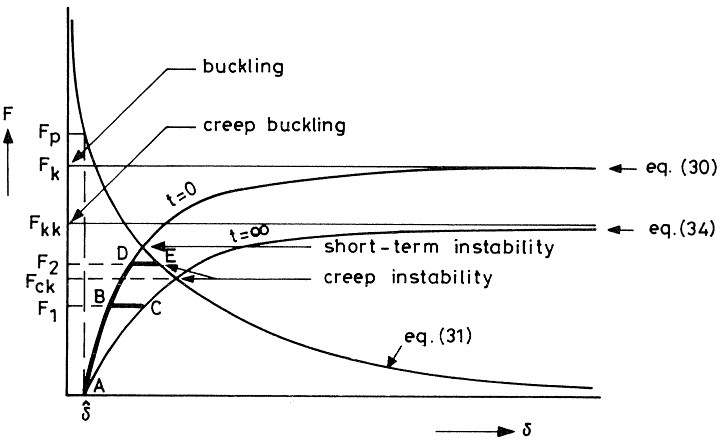


Fig. 17. Load-deflection diagram for short-term ($t=\tau$) and for long-term load ($t=\infty$).

$$\delta(\infty)=\frac{n_\infty}{n_\infty-1}\,\hat{\delta}, \qquad n_\infty=\frac{F_{k\infty}}{F} \tag{34}$$

where $F_{k\infty}$ is the so-called creep buckling load. For loads smaller than $F_{k\infty}$ the centrically (axially) loaded straight strut, after being under load for an infinite length of time, is still in stable equilibrium:

$$F_{k\infty}=\frac{k_1k_2}{k_1+k_2}\,L \tag{35}$$

The equations (35) and (34) can be derived directly from (29) and (30) by connecting in series the springs with stiffnesses k_1 and k_2, from which an "equivalent" stiffness

$$\frac{k_1 k_2}{k_1 + k_2}$$

is obtained for $t = \infty$, without further effect of the dashpot.

As seen in the diagram, the largest load that the model can support after infinitely long load duration can be determined by equation (31) and (34):

$$\frac{1}{F_{c\infty}} = \frac{1}{F_{k\infty}} + \frac{1}{F_p} \tag{36}$$

If, under constant initial load, the deflection of the strut so increases as a result of creep that the maximum permissible deflection δ_{max} according to (31) is attained, the model becomes unstable at $t < \infty$. In that case there is said to be creep instability.

Depending on the magnitude of the initial load, two different cases may occur:

1. If a load $F_1 < F_{k\infty}$ is applied at a point of time $t = \tau$, an initial deviation $\hat{\delta}$ will be increased by F_1 in accordance with (30). The short-term deviation $\delta(\tau)$ will, as a result of the creep of the rheological model, subsequently undergo a further increase to $\delta(\infty)$. Hence, on application of F_1, first a path A–B will be traversed in the load-deflection diagram and then, under constant load, the path B–C. The equilibrium of the column remains stable.
2. If the load is increased to a value $F_2 > F_{c\infty}$ at a point of time $t = \tau$, the path A–D is traversed in the load-deflection diagram. Then the deflection will, under constant load, increase until point E is reached after a time. The deflection cannot further increase under constant load, and therefore instability occurs here in consequence of creep, at a time $t < \infty$.

So far, the rheological model has been taken into account with the parameters k_1 and k_2. With $k_2 = k_1/\varphi_\infty$ the results can be expressed in the creep coefficient for load of infinite duration φ_∞, as defined in Section 3.4 of Chapter 3.

The creep buckling load $F_{k\infty}$ is then written as:

$$F_{k\infty} = \frac{F_k}{1 + \varphi_\infty} \tag{37}$$

and the long-term loadbearing capacity (i.e., the strength of the structure under sustained load) $F_{c\infty}$ as:

$$\frac{1}{F_{c\infty}} = \frac{1 + \varphi_\infty}{F_k} + \frac{1}{F_p} \tag{38}$$

The ratio of the long-term loadbearing capacity to the short-term loadbearing capacity is obtained on dividing (32) by (38):

$$\frac{F_{c\infty}}{F_c} = \frac{1 + \dfrac{F_k}{F_p}}{1 + \varphi_\infty + \dfrac{F_k}{F_p}}$$

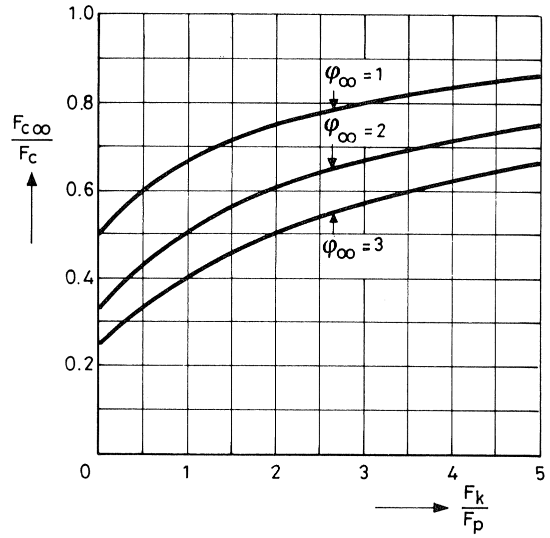


Fig. 18. Reduction of loadbearing capacity of the kinematic model by creep as a function of the ratio F_k/F_p and the creep coefficient φ_∞.

In Fig. 18 the creep sensitivity $F_{c\infty}/F_c$ as a function of the ratio F_k/F_p is represented for creep coefficients $\varphi_\infty = 1, 2$ and 3. This diagram clearly shows that the effect of creep is great if F_k is small in relation to F_p. "Translated" into an actual concrete column, this indicates high creep sensitivity of slender columns subjected to virtually centric load. Relatively short and stocky columns and/or columns under considerably eccentric load, on the other hand, are hardly sensitive to creep. Incidentally, these are conclusions which are in good agreement with what can be supposed on intuitive grounds.

4.4 *Creep buckling and creep instability with irreversible creep*

In order to investigate the effect of the choice of the rheological model, the analysis presented in the foregoing Section will be repeated here with reference to the model shown in Fig. 19. The material strain behaviour will now be schematized by the Maxwell model with time-dependent parameters as envisaged in Section 3.3. For the sake of simplicity the stiffness of the spring is assumed to be time-independent.

In Chapter 3 the constitutive equation for the Maxwell model, expressed in the creep coefficient φ, was derived. For this model the relation between a horizontal force H and a change in deformation $\mathrm{d}\delta$ is expressed by:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = \frac{1}{k}\frac{\mathrm{d}H}{\mathrm{d}t} + \frac{1}{k}\frac{\mathrm{d}\varphi}{\mathrm{d}t}H \tag{40}$$

The moment equilibrium of the model in Fig. 19 in a deflected condition is formulated by:

$$F \cdot \delta = H \cdot L \tag{41}$$

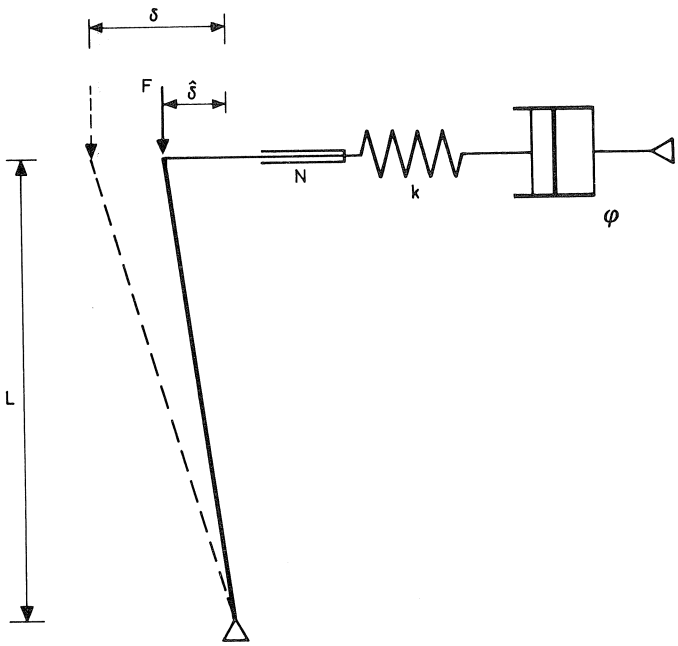


Fig. 19. Kinematic model for studying the effect of irreversible creep on the non-linear behaviour of a column.

Elimination of H from equation (40) and (41) gives:

$$\frac{1}{\delta}\frac{d\delta}{dt}=\frac{1}{n-1}\frac{d\varphi}{dt} \quad \text{where} \quad n=\frac{F_k}{F} \tag{42}$$

Finally, after integration, the following expression is arrived at:

$$\delta(t)=\delta(\tau)\exp\frac{\varphi(\tau,t)}{n-1} \tag{43}$$

With reference to this solution it can be shown that the centrically loaded straight strut is always in equilibrium so long as the load F is smaller than the buckling load F_k, for the exponent in equation (43) becomes indeterminate only for $n=1$, i.e., $F=F_k$. The creep buckling load $F_{k\infty}$ is therefore equal to the short-term buckling load F_k.

If the strut has an initial deflection $\hat{\delta}$ at the time of load application, the functional behaviour of the time-dependent deflection $\delta(t)$ is obtained from (43) by substituting:

$$\delta(\tau)=\frac{n}{n-1}\hat{\delta} \tag{44}$$

The deflection at time $t=\infty$ is obtained from (43) by substituting for $\varphi(\tau,t)$ the creep coefficient for load of infinite duration φ_∞:

$$\delta(\infty)=\frac{n}{n-1}\hat{\delta}\exp\frac{\varphi_\infty}{n-1} \tag{45}$$

Equations (44) and (45), together with the expression for the maximum possible deflection δ_{max}, equation (31), are represented graphically in Fig. 20, which in principle corresponds to Fig. 17 considered in the preceding Section.

The load corresponding to the intersection of the curves for (31) and (45) is here again defined as the long-term loadbearing capacity $F_{c\infty}$ (i.e., the strength under sustained load). Loads larger than $F_{c\infty}$ can be supported for a short time, but will in course of time give rise to creep instability.

The magnitude of $F_{c\infty}$ is found by equating (31) and (45). It is not possible, however, to arrive at an explicit expression for $F_{c\infty}$ as a function of φ_∞, F_k and F_p because this procedure yields a transcendental equation:

$$\frac{F_p}{F_{c\infty}}\hat{\delta} = \frac{n}{n-1}\hat{\delta}\exp\frac{\varphi_\infty}{n-1} \quad \text{where} \quad n = \frac{F_k}{F_{c\infty}} \tag{46}$$

A solution for $F_{c\infty}$ can be obtained without an iterative procedure as follows:

Using the auxiliary variable α, write $F_{c\infty}$ in the form:

$$\frac{1}{F_{c\infty}} = \frac{1}{F_k} + \frac{1}{\alpha F_p} \tag{47}$$

Plot a graph of

$$\frac{F_p}{F_k}\varphi_\infty$$

as a function of α as expressed by:

$$\frac{F_p}{F_k}\varphi_\infty = \frac{1}{\alpha}\ln\frac{1}{\alpha} \tag{48}$$

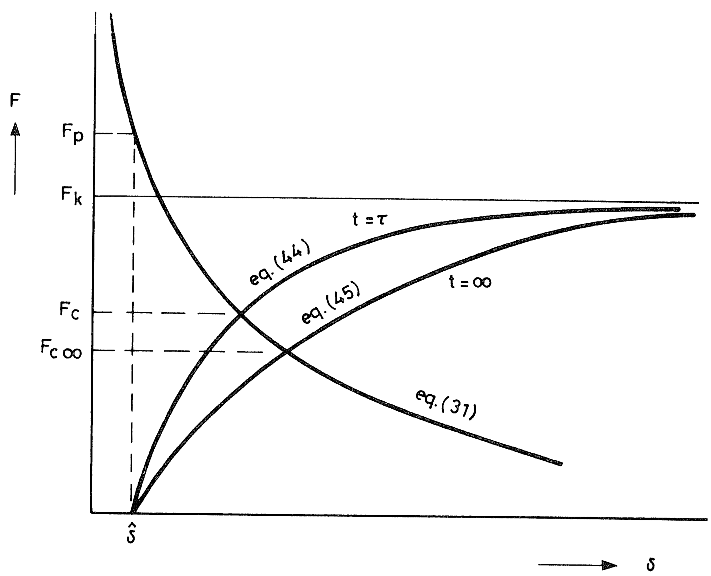


Fig. 20. Load-deflection diagram for short-term ($t = \tau$) and long-term load ($t = \infty$).

Now α can be determined as a function of

$$\frac{F_p}{F_k}\varphi_\infty$$

and $F_{c\infty}$ follows directly from equation (47).

The ratio of the long-term to the short-term loadbearing capacity is obtained as the result of dividing equation (32) by (47):

$$\frac{F_{c\infty}}{F_c} = \frac{1 + \dfrac{F_k}{F_p}}{1 + \dfrac{F_k}{\alpha F_p}} \quad \text{where } \alpha \text{ is in accordance with equation (48)} \tag{49}$$

In Fig. 21 this expression for the creep sensitivity as a function of the ratio F_k/F_p has been plotted for creep coefficients $\varphi_\infty = 1$, 2 and 3.

The curves for reversible creep, obtained in the preceding Section, have also been included in this diagram.

Appreciable differences between the two analyses are found to occur only for small values of F_k/F_p. This result is to be expected. If the buckling load is small in relation to the first-order collapse load F_p, non-linear effects will play a major part. The creep effects will be correspondingly large. According as the deflection of the kinematic model increases as a result of creep of the rheological model, the force in the latter will increase with the passage of time. Differences between rheological models manifest themselves more prominently as the loads applied to them undergo a larger change in course of time. In general, for load increasing with time, a larger deformation is calcu-

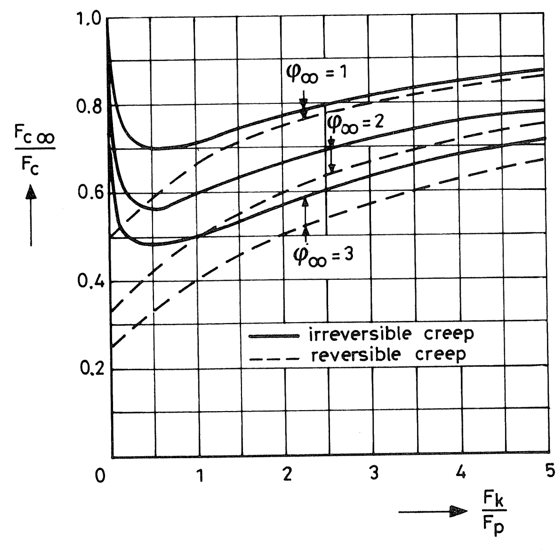


Fig. 21. Reduction of loadbearing capacity of the kinematic model by reversible creep and by irreversible creep, as a function of the ratio F_k/F_p and the creep coefficient φ_∞.

lated on the basis of reversible creep than on the basis of irreversible creep. On the other hand, with decreasing load the deformation found for reversible creep is smaller than that found for irreversible creep. In the foregoing analyses the rheological model is, as stated, subjected to load of increasing magnitude, and for this reason the creep effects are greatest in the model embodying reversible creep behaviour.

The considerations presented in this Section and in the preceding one apply to completely irreversible and to complete reversible creep respectively. These represent two extremes. The actual creep behaviour of concrete is partly reversible and partly irreversible. If the foregoing analyses are performed with a rheological model that takes account of this, a curve is obtained which is situated between the two curves found earlier on (see Fig. 21). In this case the basic assumption of completely reversible creep determines a lower bound for the long-term loadbearing capacity of the column model, whereas the assumption of completely irreversible creep gives an upper bound.

5 Steel-concrete columns

5.1 *Introduction*

The kinematic models described in the foregoing make it possible, in a simple manner, to form an idea of the effect of creep on the behaviour of a column. Strictly speaking, these models are applicable only to plain (unreinforced) concrete columns which are uncracked. The effect of steel reinforcement in a concrete section cannot be investigated with these models. A model that does allow such investigation is shown in Fig. 22. Just as in the models considered in Chapter 4, here too the entire deformation behaviour is conceived as concentrated at one point. The steel in the column is represented by two deformation elements comprising a spring and a sliding element, while the concrete is again represented by rheological models. In the model shown in Fig. 22, Dischinger's Maxwell model is employed.

5.2 *Time-dependent deformations*

The kinematic model in Fig. 22 has two degrees of freedom, namely, a compression Δ and an angular rotation expressed in a deflection δ at the top.

Starting from an initial deflection $\hat{\delta}$, the behaviour of δ and Δ as functions of time, after application of a constant centric (axial) load F, is expressed by:

$$\Delta(t) = \left\{\Delta(\tau) - \frac{F}{k_s}\right\} \exp\,(-r\varphi) + \frac{F}{k_s} \quad \text{where} \quad r = \frac{k_s}{k_c + k_s} \tag{50}$$

$$\delta(t) = \left\{\delta(\tau) - \frac{F_{ks}}{F_{ks} - F}\,\hat{\delta}\right\} \exp\,(-\mu\varphi) + \frac{F_{ks}}{F_{ks} - F} \quad \text{where} \quad \mu = \frac{F_{ks} - F}{F_k - F} \tag{51}$$

In these equations:

F_k = buckling load of the column

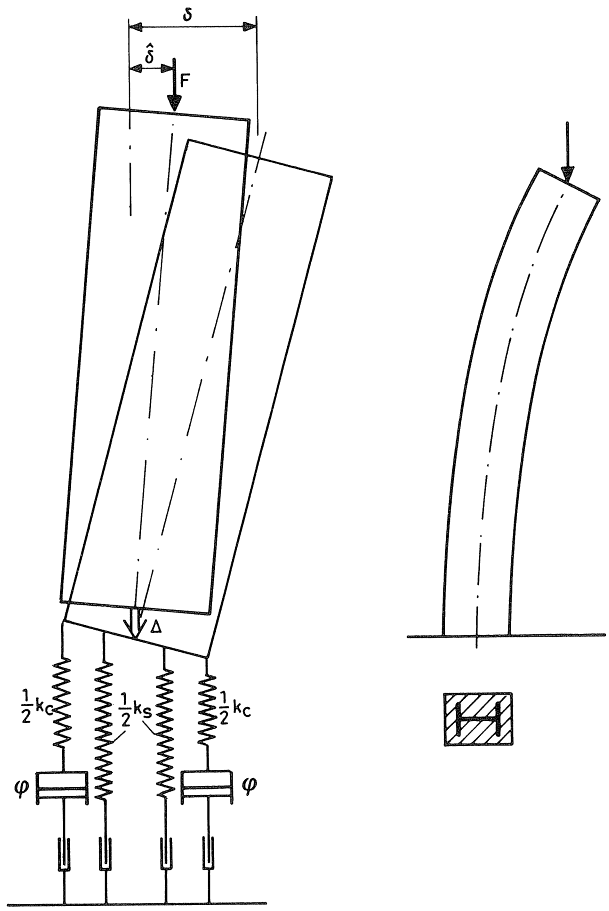


Fig. 22. Kinematic model of steel-concrete column. The steel is schematized by elasto-plastic deformation elements, the concrete by rheological models.

F_{ks} = buckling load of the column when only the stiffness of the steel is taken into account

k_c = compressive stiffness of the concrete

k_s = compressive stiffness of the steel

$\Delta(\tau)$ and $\delta(\tau)$ are the deformations directly after application of the load.

The effect of the steel in the section can be expressed in two stiffness ratios r and F_{ks}/F_k. The first is the ratio of the compressive stiffness of the steel section to that of the entire section; the second is the ratio of the flexural stiffness of the steel section to that of the entire section. The time variable t does not occur in (50) and (51). It is comprised in the creep coefficient φ, which is a function of τ and t, namely:

$$\varphi(\tau, t) = \varphi(t) - \varphi(\tau)$$

If the load F can, to begin with, be resisted elastically, then $\Delta(\tau)$ and $\delta(\tau)$ are given by:

$$\Delta(\tau) = \frac{F}{k_s + k_c}$$

$$\delta(\tau) = \frac{n}{n-1}\hat{\delta}$$

with which (50) and (51) can be rewritten as:

$$\Delta(t) = \left\{\frac{F}{k_s + k_c} - \frac{F}{k_s}\right\} \exp(-r\varphi) + \frac{F}{k_s} \tag{52}$$

$$\delta(t) = \left\{\frac{F_k}{F_k - F} - \frac{F_{ks}}{F_{ks} - F}\right\} \hat{\delta} \exp(-\mu\varphi) + \frac{F_{ks}}{F_{ks} - F} \tag{53}$$

In Fig. 23 the axial displacement Δ as a function of the stiffness ratio r is represented for creep coefficients $\varphi = 1, 2$ and 3. In Fig. 24 the relation between δ and F for various values of the flexural stiffness ratio F_{ks}/F_k has been plotted for a creep coefficient $\varphi = 1$.

In these two diagrams $\Delta(t)$ and $\delta(t)$ have been made dimensionless by dividing them by the directly occurring (elastic) deformations $\Delta(\tau)$ and $\delta(\tau)$.

In both diagrams the effect of the steel on the deformations is clearly manifest. Thus, for a creepcoefficient $\varphi = 1$, the steel brings about a reduction of 50% in the axial dis-

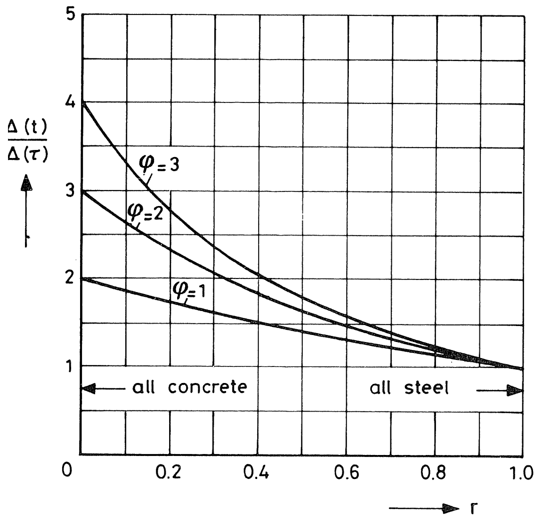


Fig. 23.
Effect of the relative quantity of steel r upon the time-dependent compressive deformation $\Delta(t)$. (r is the ratio of the compressive stiffness of the steel section to that of whole section).

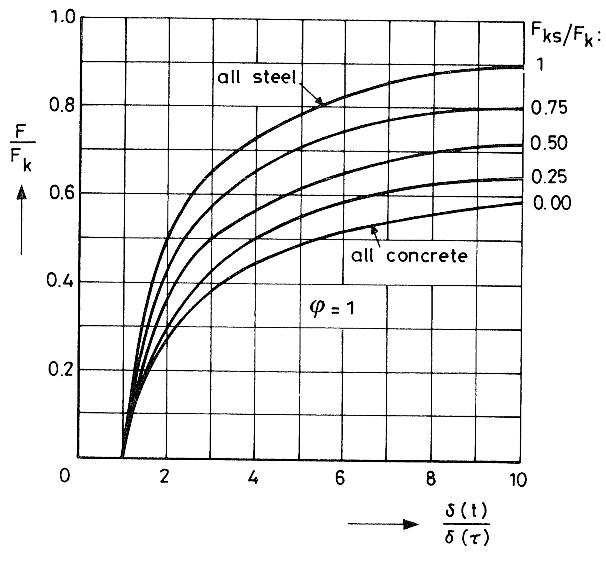


Fig. 24.
Effect of the relative quantity of steel F_{ks}/F_k upon the time-dependent deflection $\delta(t)$. (F_{ks}/F_k is the ratio of the flexural stiffness of the steel section to that of the whole section).

placement as compared with the plain concrete column, for a stiffness ratio $r = 0.4$. The reducing effect of the steel on the deflection depends not only on the flexural stiffness ratio f_{ks}/F_k, but also on the magnitude of the load F acting on the column. For a value of 0.5 or this latter ratio and a load $F = 0.5F_k$ the reduction for the reinforced with the plain concrete column is 45%. If this load is halved, the reduction percentage in this cases decreases to 18.

5.3 *Internal force distribution*

The distribution of internal forces changes in consequence of creep. As a result of increasing deformation under constant load there occurs an increase in the magnitude of the forces acting in the steel. Thus the normal force in the steel changes, as a function of time, as follows:

$$N_s(t) = k_s \Delta(t) \tag{54}$$

and the normal force in the concrete:

$$N_c(t) = F - k_s \Delta(t) \tag{55}$$

In Fig. 25 these equations (54) and (55) have been plotted qualitatively in a graph. As a result of creep there occurs a redistribution of the internal normal force in that the concrete is relieved of load and the steel is subjected to greater load.

Depending on the magnitude of the creep coefficient φ and the stiffness ratio r, this redistribution will take place to a greater or less extent. The relative increase in the normal force acting in the steel is greater according as the stiffness ratio r is smaller. The relative decrease in the normal force acting in the concrete is greater according as r is larger, i.e., if there is more steel in the section.

The internal distribution of bending moment can be deduced from equation (51). The steel moment as a function of time is:

$$M_s(t) = F_{ks}\{\delta(t) - \hat{\delta}\} \tag{56}$$

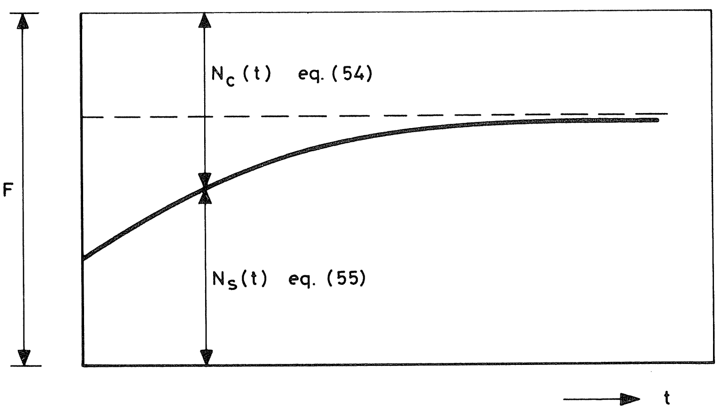


Fig. 25. Redistribution of internal normal force due to creep of concrete.

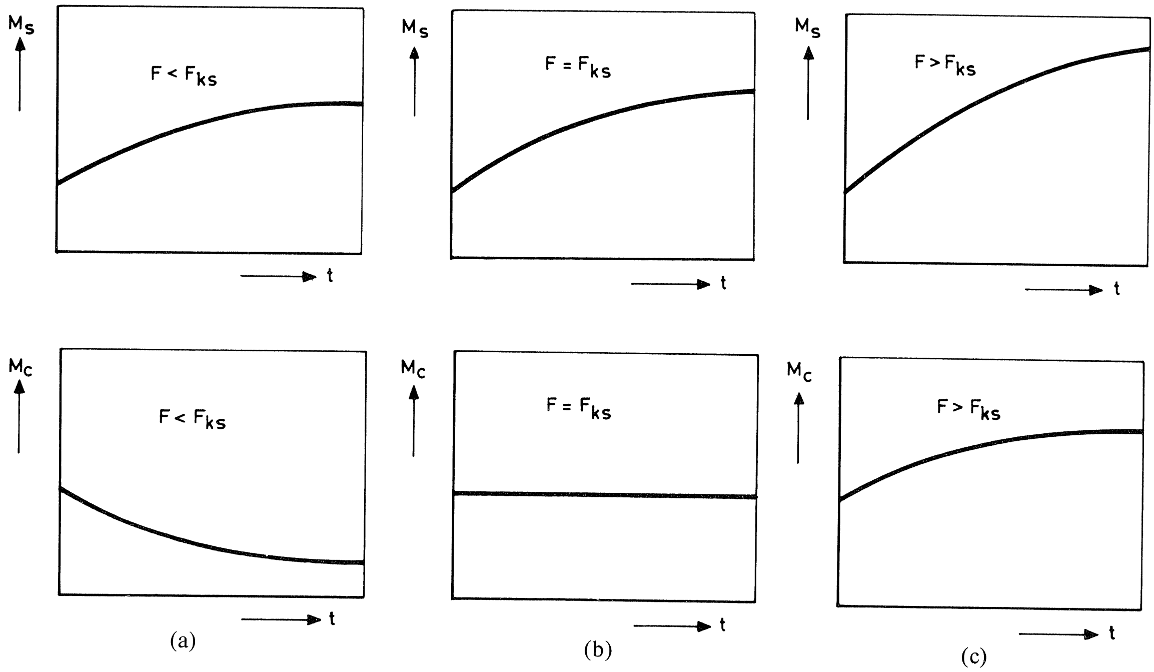


Fig. 26. Change of internal steel and concrete moments due to creep of concrete.
(a) The column load is smaller than F_{ks}. The internal concrete moment decreases due to creep.
(b) The column load is equal to F_{ks}. The internal concrete moment is constant, i.e., does not vary with time.
(c) The column load is larger than F_{ks}. The internal concrete moment, like the steel moment, increases due to creep.

and the internal moment in the concrete is:

$$\begin{aligned} M_c(t) &= F\delta(t) - F_{ks}\{\delta(t) - \hat{\delta}\} \\ &= F_{ks}\hat{\delta} + \{F - F_{ks}\}\delta(t) \end{aligned} \qquad (57)$$

With increasing deflection of the column, the steel moment increases proportionally to it. The moment exerted by the axial force F upon the column section of course also increase. Now it depends on the magnitude of F whether the moment developed by the concrete will increase or decrease in consequence of creep. If the load on the column is smaller than the buckling force of the steel F_{ks}, then, with increasing deflection, the steel moment will undergo a greater increase than will the moment acting upon the section. Therefore in this case the concrete moment will decrease in course of time. On the other hand, if the load is larger than F_{ks}, the concrete moment will, like the steel moment, increase in course of time. These relationships are qualitatively represented in graph form in Fig. 26.

5.4 *Creep sensitivity*

In the preceding Chapter it was concluded that the effect of creep on the loadbearing capacity of a column, besides depending on the creep coefficient φ, is dependent on the magnitude of the buckling load F_k in relation to the first-order collapse load F_p. Here it must be added that the quantity of reinforcing steel in the section is also an important parameter. The creep sensitivity of a reinforced concrete column can therefore be high if the creep coefficient is high, the slenderness ratio is high, the load eccentricity is low, the material is of high strength and the percentage of reinforcement is low.

These relationships are indeed very reasonable on purely intuitive grounds. The object of the present article has been to present them in a readily visualizable manner. The present treatment is, however, rather qualitative in character. In order to obtain a better idea of the quantitative effect of creep, some numerical values published by Bernard Fouré [7] will now be given. These are based on computer calculations for reinforced concrete columns. In that analysis the long-term loadbearing capacity $F_{c\infty}$ is compared with the short-term loadbearing capacity. The parameters of a standard column – such as the slenderness ratio, reinforcement percentage, etc. – are varied and the effect of this on the creep sensitivity is investigated. The standard column is of rectangular cross-section, is symmetrically reinforced (reinforcement 0.2%), and has a slenderness ratio $L/h = 25$. The load acts with constant eccentricity $e_0 = 0.1h$, while the creep coefficient is $\varphi_\infty = 2$.

On varying the respective parameters the following results are obtained:

Variation of the slenderness ratio L/h

L/h	10	15	20	25	30	40	50
$\frac{F_{c\infty}}{F_c}$	0.88	0.77	0.69	0.65	0.58	0.52	0.50

Variation of the eccentricity e_0

e_0/h	0.03	0.05	0.10	0.20	0.30	0.50	1.00
$\frac{F_{c\infty}}{F_c}$	0.66	0.65	0.65	0.72	0.79	0.84	0.90

Variation of the reinforcement percentage ω

ω%	0	0.05	0.1	0.2	0.3	0.6	1.0
$\frac{F_{c\infty}}{F_c}$	0.46	0.51	0.57	0.65	0.71	0.78	0.82

Variation of the creep coefficient φ_∞

φ_∞	0	0.5	1.0	2.0	3.0	4.0
$\frac{F_{c\infty}}{F_c}$	1.00	0.87	0.78	0.65	0.57	0.51

The standard column in this example has only a low reinforcement percentage. From the table given above, however, it is apparent that even this reinforcement brings about a substantial reduction in creep sensitivity as compared with the plain concrete column. It further emerges from these tables that even a low value of the creep coefficient φ_∞ causes a relatively large decrease in the loadbearing capacity of the column. The effect of the magnitude of the eccentricity e_0 is practically constant for values of e_0 smaller than $0.1h$. If e_0 is increased, there is a rapid decrease in creep sensitivity, and if e_0 is of the order of $2h$ to $3h$, the creep of the concrete plays practically no part in determining the magnitude of the loadbearing capacity of the column.

6 Summary and conclusions

The effect of creep of concrete on the physical and geometrical non-linear behaviour of (reinforced) concrete columns can be analysed in a suitably visualizable manner with the aid of a combination of a kinematic and a rheological model. At first sight such a model shows little resemblance to an actual column. The mechanical behaviour of the model under short-term and under long-term (i.e., sustained) load can, however, be expressed in quantities that are applicable to an actual column.

These quantities are successively:

- The Euler buckling load, defined as the (centric) load in excess of which no stable equilibrium is possible.
- The first-order collapse load, defined as the collapse load obtained when no second-order effects are taken into account.
- The reinforcing steel percentage.
- The creep coefficient, defined as the ratio of the creep strain of a concrete fibre under constant stress and the directly occurring strain.

With the aid of the models presented in this article, and by means of a simple approach, the following conclusions as to the short-term and the long-term loadbearing capacity of a column are arrived at. In this context two different cases are to be distinguished:

1. If the Euler buckling load of a column is large in relation to the first-order collapse load, second-order effects will pay a minor part and need not be taken into account in the analysis. In that case the deformations due to creep have no effect on the loadbearing capacity of the column either.
2. On the other hand, if the Euler buckling load is only a few times as large as the first-order collapse load (or even smaller), a non-linear method of analysis should be used for the column. In consequence of creep the loadbearing capacity may, depending on the magnitude of the creep coefficient, undergo a considerable reduction.

The first case relates to short and stocky columns or to columns under highly eccentric load. The second relates to slender columns with high material strength which are under predominantly centric (axial) load. The creep sensitivity of such columns can be modified in a favourable sense by the incorporation of extra steel reinforcement.

With the models presented in this article it is possible in a relatively simple manner to derive expressions for the time-dependent deflections and shortening deformations.

These results are quantitatively applicable to actual (reinforced) concrete columns.

The mathematical formulation of the creep of concrete calls for the following comments. The creep behaviour can be characterized by a family of creep functions. Time-dependent deformations of concrete structures can be calculated on the basis of these creep functions and by applying the principle of superposition. A calculation procedure of this kind leads to integral equations which are not amenable to mathematical treatment, though analytical solutions are possible in special cases. In general, however, numerical methods of solving them will have to be used.

If a more conveniently visualizable analytical approximation is desired, rheological models can successfully be employed. These lead to differential equations which are more readily accessible to mathematical treatment. Essentially the choice of a rheological model amounts to choosing a family of creep functions of special mathematical form. The latter is determined by the structure of the model. In this sense, therefore, a method of analysis which is based on a rheological model is less general than one which is based on arbitrarily chosen creep functions. This need not, however, be an obstacle to practical calculations, considering that there is an infinite number of possibilities in constructing a rheological model.

With a rheological model a mathematical formulation of creep behaviour is presented in a form that can be visualized, so that insight into the structural consequences of creep is improved. This of course can provide an important argument in favour of using such models.

Two methods of analysis well known in the literature are based on rheological models, namely, the "effective modulus method" and the "rate of creep method". In the first of these two methods the creep of concrete is assumed to be completely reversible, whereas in the second it is assumed to be completely irreversible. Although both these approaches are no more than approximations of the actual creep behaviour of concrete, quite serviceable creep calculations can be done with both of them. The advantage of the "effective modulus method" is that it can provide a good idea of the creep sensitivity of a concrete structure on the basis of a simple calculation. Its simplicity is due to the straightforward reduction applied to the stiffness of the concrete, so that creep is conceived as occurring directly. The "rate of creep method" involves solving a first-order differential equation. Even so, it is still a relatively simple method, and the creep deformations calculated in this way are in general amply accurate for practical purposes.

If the concrete stresses in a structure increase in course of time as a result of creep, an upper bound for the time-dependent deformations is found with the aid of the "effective modulus method". If the concrete stresses decrease in course of time, the "rate of creep method" enables an upper bound for the creep deformations to be determined. Concrete stresses increasing with time may occur, inter alia, in slender concrete columns containing a low percentage of reinforcement (less than about 2%); concrete stresses decreasing with time may occur, inter alia, in columns with a high reinforcement percentage and in prestressed concrete beams.

The creep formulations contained in the CEB-FIP guidelines of 1970 and 1978 are likewise based on rheological models. In these models the combined reversible-irreversible creep behaviour of concrete has been taken into account. In principle, accurate creep calculations can be done with these models, but of course it is of major importance to introduce correctly estimated creep coefficients. It is often advisable to do several creep calculations into which minimum and maximum anticipated creep values are successively introduced.

7 References

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