Geometrically non-linear behaviour of a bar with shear deformation

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1 Introduction

This article consists of two parts. The first part (section 2) deals with the bar without bending deformation, in which only shear deformation occurs. Such a bar may serve as a simplified model for the frame in a high-rise building. To investigate the properties required of the model we will look in more detail at the behaviour of a single storey taken out of the frame.

The second part (section 3) deals with the bar in which both shear deformation and bending deformation occur. The influence of the shear rigidity on the buckling load of a simply supported bar is investigated.

2 The bar with shear deformation only

Kinematic and constitutive relations

The frame in Fig. 1 is assumed to have rigid beams for the sake of simplicity. Those beams may be seen as the planes of the cross-sections in a bar serving as a simplified model for the frame. With columns that have an infinite extensional rigidity, these cross-sections can only shift but not rotate with respect to each other. Each storey of the frame may be seen as a shear-element. With a sufficiently large number of storeys recourse may be taken to the continuous model of a bar with shear deformation only [1].

The shear deformation γ consists of the rotation φ of the cross-section d and the slope w' of the bar axis s, expressed by the (kinematic) relation (Fig. 2):

$$\gamma = \varphi + w'$$

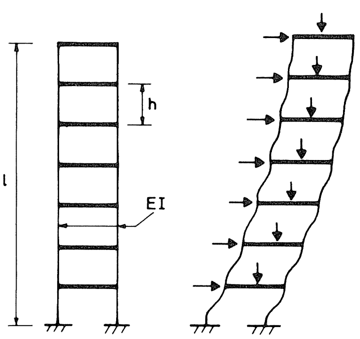


Fig. 1.

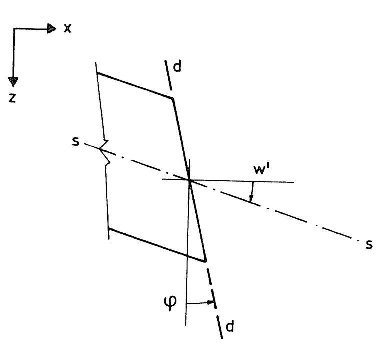


Fig. 2.

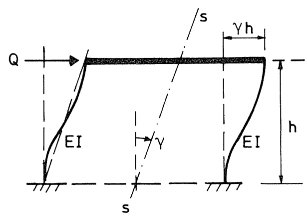


Fig. 3.

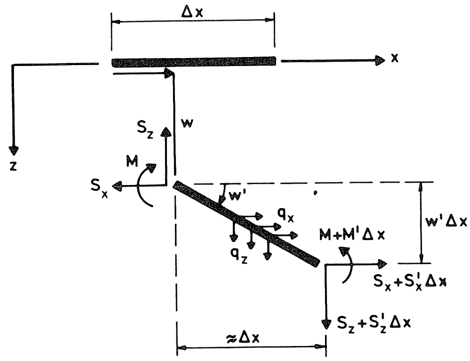


Fig. 4.

The shear deformation is caused by the (transverse) shear force Q. The (constitutive) relation between Q and γ is:

$$Q = K\gamma = K(\varphi + w')$$

For the frame in Fig. 1 the shear rigidity K is (Fig. 3):

$$K = \frac{Q}{\gamma} = 24\,\frac{EI}{h^2}$$

Equilibrium relations

In a geometrically non-linear calculation the equilibrium equations refer to the deformed bar. In Fig. 4 an elementary part of the deformed bar is shown. For the present the cartesian components S_x and S_z of the stress resultant S are used; later on S will be resolved into an extension force N and a shear force Q, in a way that is not trivial. The equilibrium of the element in Fig. 4 results in the following three relations:

$$S_x' + q_x = 0 \tag{1}$$

$$S_z' + q_z = 0 \tag{2}$$

$$M' - S_x w' + S_z = 0 \tag{3}$$

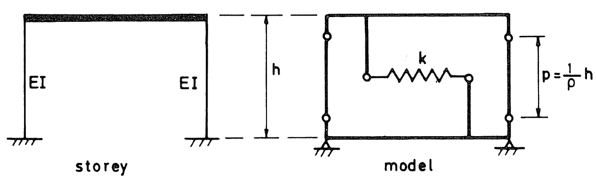


Fig. 5.

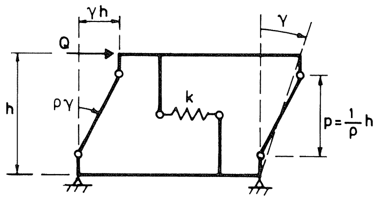


Fig. 6.

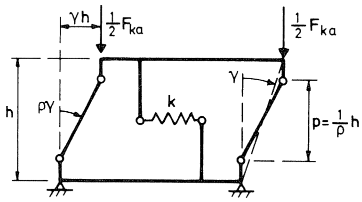


Fig. 7.

A model for the shear element

To explain the geometrically non-linear behaviour of a shear-element (a single storey in the frame of Fig. 1) the model in Fig. 5 is introduced. According to its definition the shear force causes the shear deformation and for that reason the shear force equals the force in the spring. The shear rigidity K of the model is (Fig. 6):

$$K=\frac{Q}{\gamma}=\frac{k\gamma h}{\gamma}=kh$$

The shear rigidity is independent of the length p of the hinged column segments. The buckling force of the shear element is the (smallest) value of F for which a state of equilibrium is possible in a deformed configuration (Fig. 7). In that case the horizontal component of the sloping forces in the hinged column segments has to be in equilibrium with the force in the spring:

$$\varrho\gamma F_{\text{ka}}=Q=K\gamma\rightarrow F_{\text{ka}}=\frac{1}{\varrho}K=kp$$

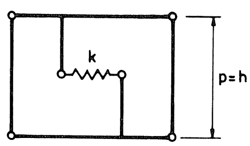
The buckling force of the shear-element depends on the length p of the hinged column segments, and is independent of the height h of the element.

The characteristic properties K and F_{ka} of the shear-element may be obtained by an appropriate choice of the spring stiffness K and the ratio $\varrho=h/p$ in the model.

If $\varrho=1$ then the length of the hinged column segments equals the height of the elements (Fig. 8) and the buckling force equals the shear rigidity.

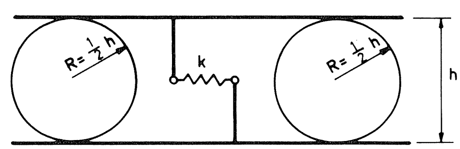
If $\varrho=0$ then the hinged column segments have an infinite length and operate in the same way as rollers with a radius $R=h/2$ (Fig. 9). The buckling force is infinite.

For values of ϱ between 0 and 1 the buckling force is larger than the shear rigidity.



$\rho=1$; $F_{ka}=K$

Fig. 8.



$\rho=0$; $F_{ka}=\infty$

Fig. 9.

For the frame in Fig. 1, with rigid beams, the following relation holds:

$$\varrho = \frac{K}{F_{ka}} = \frac{24EI/h^2}{2\pi^2 EI/h^2} = 1.22$$

When the beams are not rigid, the value of ϱ is between 1 and 1.22 and in that case the buckling force is smaller then the shear rigidity.

The direction of the shear force and the extension force

In the case of a geometrically non-linear approach to problems with shear deformation, one needs to know more about the changing directions of the shear force Q and extension force N. Therefore the model of the shear-element is investigated in the deformed state (Fig. 10).

In response to the shear deformation, the shear force Q has the direction of the spring in the model. Thus the shear force remains active in the plane of the cross-section.

It will be clear that the direction of N is the same as the direction ϑ of the hinged column segments in the deformed state. From Fig. 10 it may be concluded that:

$$\vartheta = w' + (\varrho - 1)\gamma = \varrho w' + (\varrho - 1)\varphi$$

The extension force appears to be inclined at an angle of $(\varrho - 1)\gamma$ with respect to the axis.

The direction of N, and not the direction of Q, depends on ϱ. Only when $\varrho = 1$(Fig. 8) the direction of N coincides with the bar axis (Fig. 11). Due to the deformation of the bar, N and Q are not in general perpendicular to each other (Fig. 10). The initially right angle is diminished by the amount $\varrho\gamma$ (a positive value of $\varrho\gamma$ means the angle decreases).

When $\varrho = 0$ (Fig. 9), N and Q remain perpendicular to each other; the extension force works at a right angle to the plane of the cross-section and is inclined at an angle γ to the bar axis (Fig. 12).

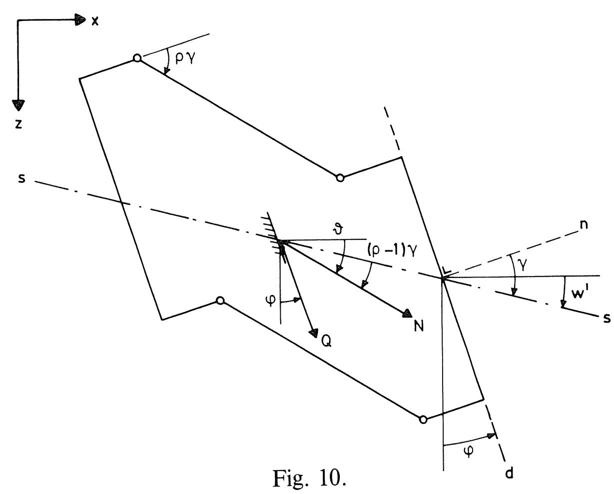


Fig. 10.

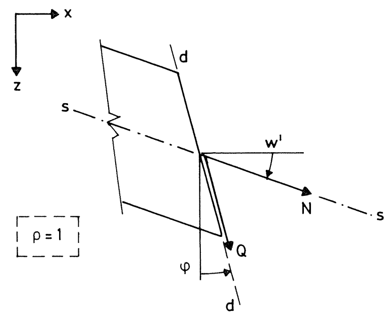


Fig. 11.

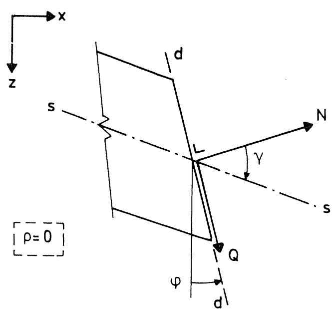


Fig. 12.

When $\vartheta \ll 1$ and $\varphi \ll 1$ then the components S_x and S_z of the stress resultant S are (Fig. 13):

$$S_x = N + Q\varphi \tag{4}$$

$$S_z = Q + N\vartheta \tag{5}$$

Differential equations

The equilibrium equations may be expressed with respect to N and Q by substitution of (4) and (5) in (1), (2) and (3). After combining some terms and neglecting the terms with products of ϑ, φ and w', the result is:

$$N' + Q\varphi' + q_x - \varphi q_z = 0 \tag{6}$$

$$Q' + N\vartheta' + q_z - \vartheta q_x = 0 \tag{7}$$

$$M' - N(\vartheta - w') - Q = 0 \tag{8}$$

Without bending deformation, all cross-sections remain parallel, so $\varphi' = 0$ holds. With $Q = K(\varphi + w')$ and $\vartheta = \varrho w' + (\varrho - 1)\varphi$ the relations mentioned above simplify to:

$$N' + q_x = 0 \tag{9}$$

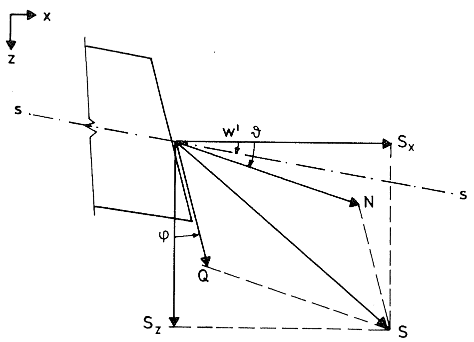


Fig. 13.

$$(K+\varrho N)w''+q_z=0 \tag{10}$$

$$M'-\{K+(\varrho-1)N\}(\varphi+w')=0 \tag{11}$$

The last equation is required if one has to determine the bending moment in the bar.

Two buckling problems

The geometrically non-linear behaviour will be investigated for the two simply supported bars in Fig. 14. Both bars can deform by shear only and have the same shear rigidity K.

The bar in Fig. 14a is composed of shear-elements according to the model in Fig. 8 for which $\varrho=1$ applies.

The bar in Fig. 14b is composed of rigid rectangular blocks, linked by slip connectors. In the slip direction, forces can be transmitted by springs. Each shear-element has an infinitely large buckling load, thus $\varrho=K/F_{ka}=0$ (see also the model in Fig. 9).

With $q_x=0$ and $q_z=0$ the equations (9) and (10) may be simplified:

$$N'=0 \tag{12}$$

$$(K+\varrho N)w''=0 \tag{13}$$

The first equation means that N is constant. Now the question is: for which value of N the state of equilibrium will become unstable, because equilibrium is also possible in a deformed configuration.

Equation (13) is fulfilled for $\varrho=1$ (Fig. 14a) when $N=-K$ holds. The buckling force is a compressive force and equals the shear rigidity. The buckling form is undetermined and the cross-sections do not rotate ($\varphi=0$). A possible buckling form is given in Fig. 15a.

Equation (13) can be fulfilled for $\varrho=0$ (Fig. 14b) only when the axis remains straight. Thus $w=0$ for $0\leqslant x\leqslant l$. In that case follows from (11):

$$M=(K-N)\varphi x+C_1$$

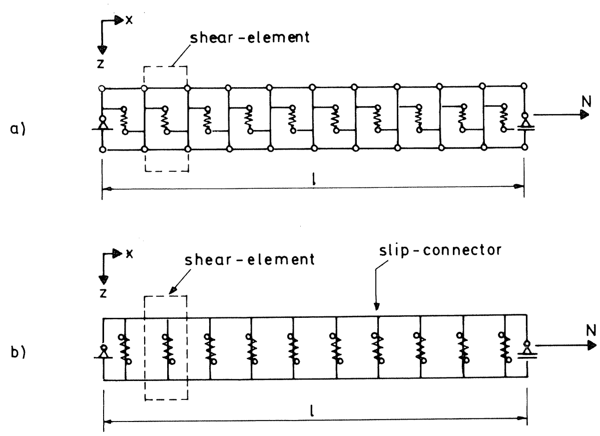


Fig. 14.

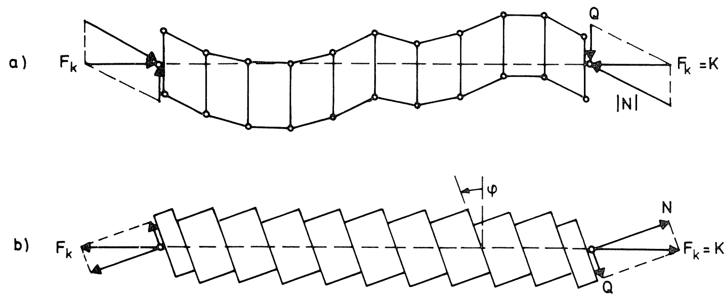


Fig. 15.

Because $M=0$ in $x=0$ and $x=l$, this equation can only be fulfilled when $N=K$ holds. The buckling load appears to be a tensile force. The buckling form is given in Fig. 15b.

Both bars in Fig. 14 with shear deformation have the same geometrically linear behaviour, but the geometrically non-linear behaviour is quite different!

Unbraced frames (continuous model)

For the frame in Fig. 16, represented by the model of a bar with only shear deformation, one has $\varphi=0$ and thus $\vartheta=\varrho w'$. When $q_x=0$ and N is a constant compressive force F, then the differential equation governing the problem is:

$$(K-\varrho F)w''+q_z=0 \tag{14}$$

Two boundary conditions are available:

$$x=0;\ w=0$$

$$x=l;\ Q=S_z-\vartheta S_x=0+\varrho F w'\rightarrow(K-\varrho F)w'=0$$

Suppose w_1 is the solution of (14) in the case $F=0$. In other words w_1 is the so-called first order displacement. The solution w_2 of (14) for the case $F\neq 0$ may be written in the following form:

$$w_2=\frac{K}{K-\varrho F}w_1=\frac{n}{n-1}\,w_1 \text{ where } n=\frac{F_{ka}}{F}=\frac{K}{\varrho F}$$

The so-called second order displacement w_2 is proportional to the first order displacement w_1, and in their relation the well known multiplication factor $n/(n-1)$ appears.

Unbraced frames (discrete model)

Generally the extension force in the model for the frame will not be constant and also the shear rigidity may vary along the height of the frame.

For each storey one has (Fig. 17):

$$\Delta w_2=\frac{K}{K-\varrho F}\Delta w_1$$

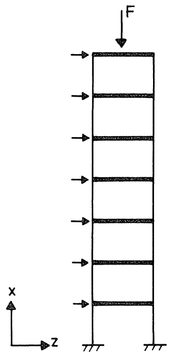


Fig. 16.

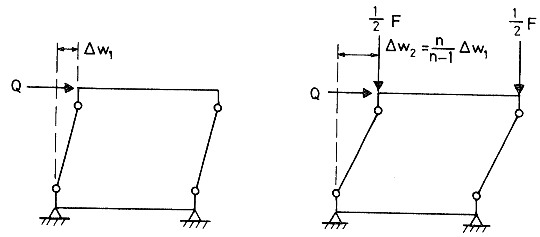


Fig. 17.

where F is the compressive force to be transmitted by the storey.

Using the results of a linear calculation for each storey the shear rigidity K may be calculated from $Q = S_z$ and Δw_1:

$$K = \frac{Q}{\gamma} = S_z \frac{h}{\Delta w_1}$$

where h is the height of the storey. Now it is possible to write:

$$\Delta w_2 = \frac{1}{1 - \varrho \dfrac{F}{S_z} \dfrac{\Delta w_1}{h}} \Delta w_1 \tag{15}$$

All properties are taken with reference to the same storey. The values of F, S_z, h and Δw_1 are known. The magnitude of ϱ, lying between 1 and 1.22, has to be estimated. When $F \ll F_{ka}$ then the influence of ϱ in (15) is small. Formula (15) may be considered as a simple and efficient tool to get information about the second order displacements of an unbraced frame.

3 The bar with shear and bending deformation

Differential equations

In fig. 18 the model for a shear-element has been changed into a shear-bending-element. The earlier discussion about the directions of N and Q remains valid without alteration. For the continuous model, i.e. the bar with shear and bending deformation, the constitutive relations are:

$$M = EI\varphi'$$

$$Q = K(\varphi + w')$$

Substitution of these relations in the equilibrium equations (6), (7) and (8) results in:

$$N' + q_x = 0 \tag{16}$$

$$K(\varphi' + w'') + N\{\varrho w'' + (\varrho - 1)\varphi'\} + q_z = 0 \tag{17}$$

$$EI\varphi'' - \{K + (\varrho - 1)N\}(\varphi + w') = 0 \tag{18}$$

A buckling problem

The buckling problem concerns a bar with a constant compressive force $N = -F(f_x = 0; f_z = 0)$. The specification of support conditions is postponed. For this bar the equations (17) and (18) may be written in the following form:

$$\begin{bmatrix} \{K - (\varrho - 1)F\} - EID^2 & \{K - (\varrho - 1)F\}D \\ \{K - (\varrho - 1)F\}D & \{K - \varrho F\}D^2 \end{bmatrix} \begin{bmatrix} \varphi \\ w \end{bmatrix} = 0 \tag{19}$$

D is a differential operator: $D(\ldots) = d(\ldots)/dx$.

Suppose (19) has a non-trivial solution of the form:

$$\begin{bmatrix} \varphi \\ w \end{bmatrix} = e^{sx} \begin{bmatrix} \Phi \\ W \end{bmatrix}$$

Substitution in (19) leads to two homogeneous linear algebraic equations with Φ and W as unknowns. A non-trivial solution exists when the determinant is zero, resulting in the condition:

$$(K - \varrho F)EIs^4 + F\{K - (\varrho - 1)F\}s^2 = 0$$

The four roots are:

$$s_1 = s_2 = 0$$

$$s_3 = -s_4 = s \text{ where } s = \sqrt{-\frac{F}{EI}\left\{1 + \frac{F}{K - \varrho F}\right\}}$$

The question whether s is real or imaginary remains open for the time being. The buckling form is given by:

$$w = C_1 + C_2 x + C_3 e^{sx} + C_4 e^{-sx}$$

$$\varphi = -C_2 + \left\{\frac{F}{K - (\varrho - 1)F} - 1\right\}(C_3 e^{sx} - C_4 e^{-sx})s$$

For increasing K the influence of ϱ decreases. When $K = \infty$ and only bending deformation occurs, then $s = \sqrt{-EI}$ and $\varphi = -w'$ apply.

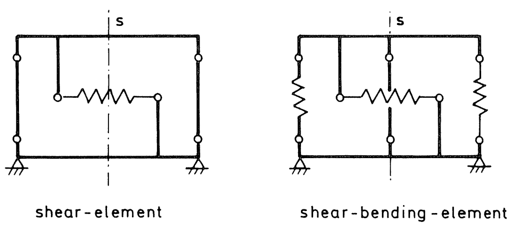


Fig. 18.

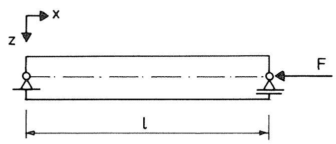


Fig. 19.

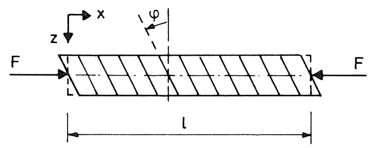


Fig. 20.

The buckling problem is investigated in more detail for a bar that is simply supported in both ends (Fig. 19). The boundary conditions are:

$$w = 0 \text{ and } M = EI\varphi' = 0 \text{ in } x = 0 \text{ and } x = l.$$

These conditions lead to four homogeneous linear algebraic equations with the constants C_1, C_2, C_3 and C_4 as unknowns. A non-trivial solution exists when the determinant equals zero:

$$Det = F^2(e^{sl} - e^{-sl}) = 0 \qquad (20)$$

Supposing for one instant that s is real, one obtains $s = 0$ as the single solution to this equation. Then the buckling force is:

$$F_k = \frac{1}{\varrho - 1} K \qquad (21)$$

When $\varrho > 1$ then F_k is a compressive force, but when $\varrho < 1$ then it is a tensile force. The matching buckling form is drawn in Fig. 20. The bar deforms only by shear and not by bending.

When s is imaginary however, then (20) has more than one solution. Taking $s = \alpha i$, condition (20) changes into:

$$e^{sl} - e^{-sl} = 2i \sin \alpha l = 0$$

with the solutions $\alpha l = 0, \pi, \ldots$ in which

$$\alpha = \sqrt{\frac{F}{EI}\left\{1 + \frac{F}{K - \varrho F}\right\}}$$

The solution $\alpha l = 0$ agrees with the value in (21).
For $\alpha l = \pi$ the value of F_k is the root of a quadratic equation:

$$(\varrho - 1)F^2 - (K + \varrho F_E)F + KF_E = 0 \text{ where } F_E = \frac{\pi^2 EI}{l^2} \qquad (22)$$

The buckling form is given by:

$$w = C \sin \alpha x$$

$$\varphi = -\frac{K - \varrho F}{K - (\varrho - 1)F} \alpha C \cos \alpha x$$

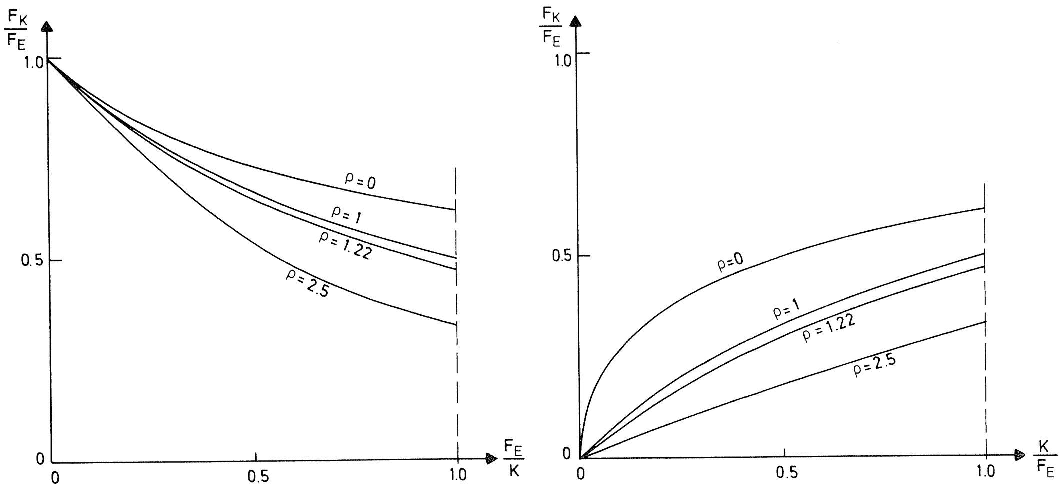


Fig. 21.

When $\varrho = 1$, then (22) has a single solution that can be written in the following form:

$$\frac{1}{F_k} = \frac{1}{F_E} + \frac{1}{K}$$

When $\varrho \neq 1$, then (22) always has two roots. When $\varrho > 1$ both roots are positive, the smallest root of which gives the compressive buckling force. This force is smaller than the compressive force according to (21) and is therefore the critical value.

When $\varrho < 1$ then the roots have a different sign. Not only a compressive buckling force exists (the positive root), but also a tensile buckling load (the negative root). In the case of tension the buckling load according to (21) gives the critical value, however.

In Fig. 21 the compressive buckling load F_k is given for several values of ϱ as a function of F_E/K (or K/F_E). This diagram shows the influence of the shear rigidity on the buckling load. In Fig. 22 the same influence is presented in another way.

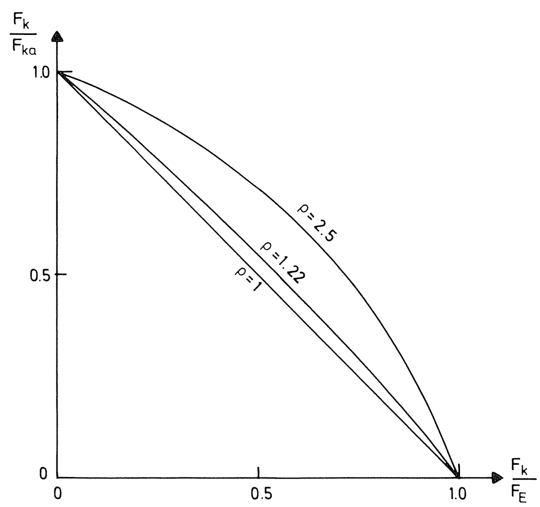


Fig. 22.

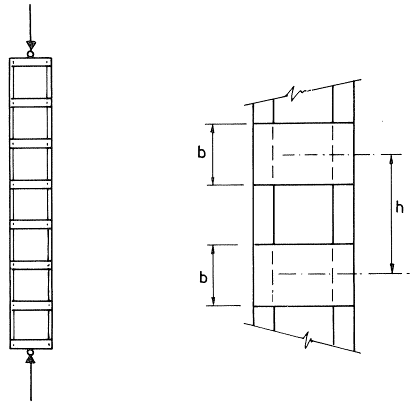


Fig. 23.

In [2] the influence of the shear deformation is treated in two different ways, with results that agree with the particular cases $\varrho = 0$ and $\varrho = 1$.

The line with $\varrho = 2.5$ (taken as an example) is suitable for the built-up bar in Fig. 23, with large and very stiff coupling plates, that cannot reasonably be modelled as line elements. The case of a built-up bar is also treated in [2], but the equation for the buckling load is given in such a form that it can only be dealt with iteratively. This seems unnecessarily roundabout, but the results appear to be the same as those from the formula presented here.

4 Conclusion

It is surprising that such a simple model for a shear-element provides so much more insight in the fundamental behaviour of a beam with shear deformation. Some of the problems presented and worked out in this paper are copied from the studies [3] and [4]. Together with [5], in which the geometrically non-linear behaviour of a high-rise frame linked to a core is investigated, these studies are inspired by the lectures of prof. ir. A. L. Bouma.

5 Symbols

EI bending rigidity
F_k buckling force of the whole bar
F_{ka} buckling force of a single shear-element
K shear rigidity
M bending moment
N extension force (in the context of this article the more usual name of normal force would be inappropriate)
Q shear force
q distributed loading
S stress resultant

w displacement in z-direction

x, z Cartesian coordinates, with the x-coordinate along the bar axis (differentiation with respect to x is denoted by an apostrophe)

γ shear deformation

ϑ $= \varrho w' + (\varrho - 1)\varphi$

ϱ $= K/F_{ka}$

φ rotation of the plane of a cross-section

6 Literature

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