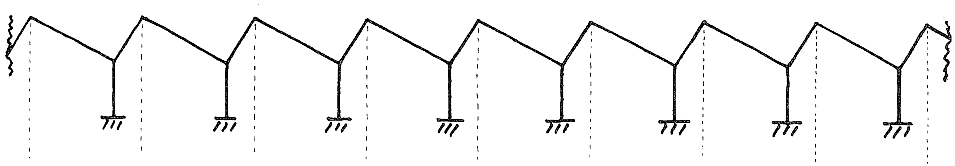
Heron's Fountain

Communications concerning novel conceptions and ideas in which the surprising element has something in common with the playful inventions of Heron of Alexandria, after whom this journal is named.

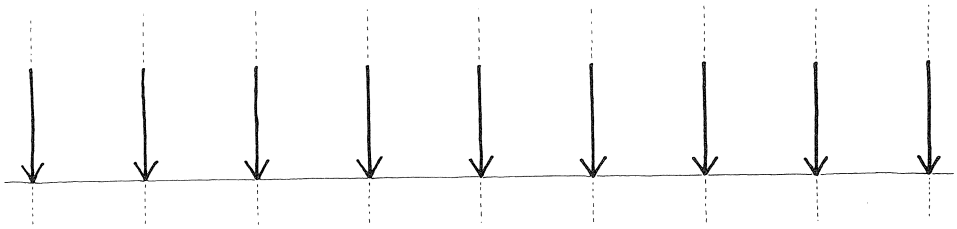
Frames with fearful symmetry

by I. M. MORTELHAND

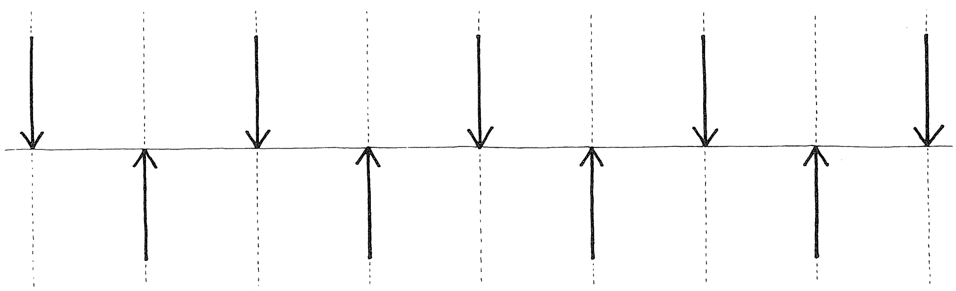
Symmetry considerations in structural mechanics usually refer to either mirror symmetry or axial symmetry. In geometry many other possibilities are mentioned [1], which merit some attention of at least an exploratory nature. The present article deals by way of example with a structure that exhibits translation symmetry. The north-light roof with many bays, as shown in the figure, will coincide point for point with itself after translation over a distance equal to the length of one bay. This is of course only true in a strict sense if the structure extends to infinity on both sides. For practical applicability of symmetry considerations to the bays that are centrally located it is sufficient that the mechanical phenomena at the far edges are local effects, not noticeable beyond a few bays. This is certainly true here, because the columns are firmly fixed to the foundation, to which any loading can be transferred.



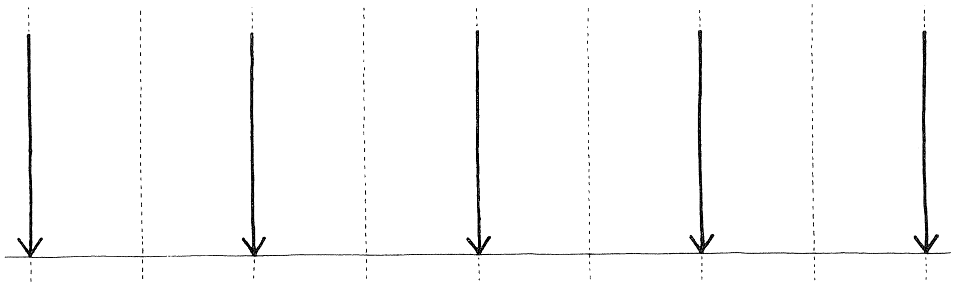
A matter of greater concern is whether all the properties and quantities that determine mechanical behaviour, share in the translation symmetry of the geometrical form of the structure. One may reasonably assume that the dimensions of members, material properties, and degrees of fixity in joints and supports are the same for all the bays. The loading on the bays however, will be different in some of the cases that are relevant to structural design. In a structure under variable loads the largest stress often occurs for non-uniform loading, e.g. when alternating bays are loaded and load-free. In such a case one will have to take recourse to subtle strategies in order to enjoy the benefits that follow from the use of symmetry considerations. Prior to the treatment of this problem (and still more complicated cases) a simple example is considered, with a kind of loading which does in fact exhibit the same symmetry as the structure. It consists of concentrated forces with the same magnitude and direction, whose lines of action pass through the joint at the top of the roof. The load-diagram contains these lines; the drawing of the structure is not repeated.



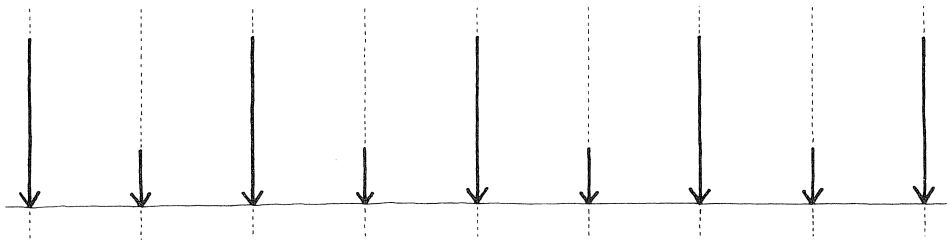
In order to demonstrate a possible use of symmetry considerations, let it be assumed that extensional deformation of the members is taken into account; the analysis will proceed according to the displacement method. The deformation pattern will exhibit the same symmetry as the structure and the loading. For each bay the joint displacements must be the same. Since the number of unknowns for a plane frame is three times the number of independent joints (for each joint two translational components and a rotation have to be determined) the total number of unknowns is six. It is still conceivable to solve the equations by hand, although of course in this day and age one may not care to do so, given the availability of cheap computing power. Not many computer programmes deal directly with translation symmetry, so setting up the equations requires human help.



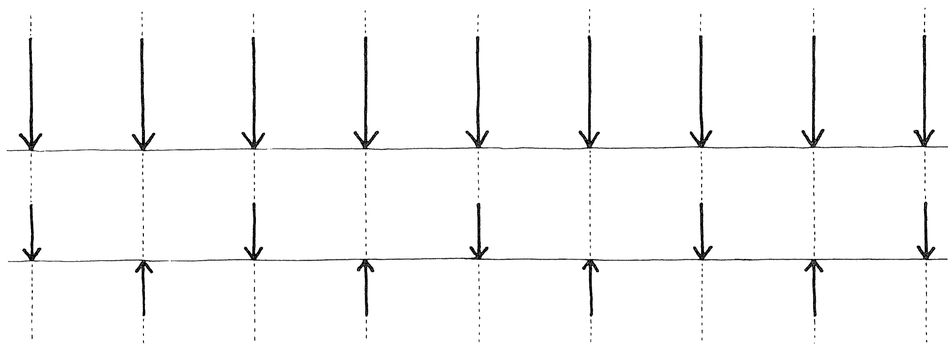
The case shown in the load-diagram above becomes the opposite of itself when a translation over the distance of one bay length is applied. This case is called contra-symmetric (and the preceding case co-symmetric), a comparison with the translation symmetry of the structure being implied. Explicitly stating that the calculation proceeds in the domain of linear mechanics, where in particular the strains due to compressive stress are exactly the opposite of strains due to tensile stress, one arrives at the conclusion that the deformation pattern will be contra-symmetric. Displacements in consecutive bays will be opposite to each other. The calculation according to the displacement method can be restricted to one bay and needs to deal only with six unknowns.



The case of alternating bays with and without load has been mentioned at an earlier stage; now is the time to deal with it. Quite simply one can superimpose the results of the previous co- and contra-symmetric cases, and a case according to the diagram above will appear. Conversely, if such a case is given in the first place, one can split it up as follows: halve the loading on the bay that is loaded, put this on all bays, which yields the co-symmetric part; then put plus and minus the halved loading on consecutive bays, in order to obtain the contra-symmetric part.



A slightly different case occurs when alternating bays carry a large and small load. Now the co-symmetric part is found by taking the average of the two loads, and the contra-symmetric part as the remainder after the average is subtracted. Again the words co-symmetric and contra-symmetric refer to the symmetry of the structure for translation over a distance equal to the length of one bay.

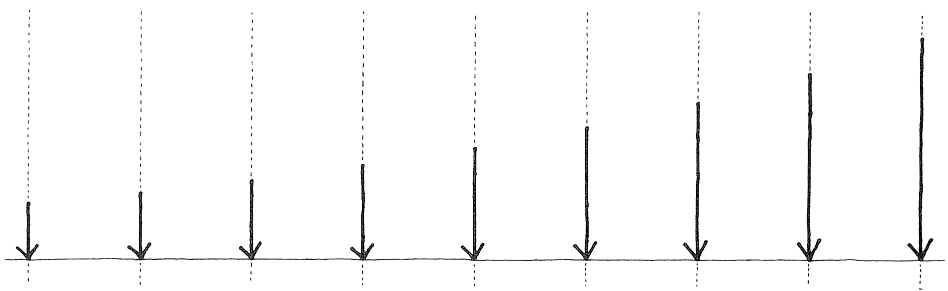


This last result has a rather weak claim to the name of generalization. As a preliminary to broader discussion, a simplified terminology is now introduced. The basic symmetry operation of the structure will be called a one-bay shift. Of course also two-bay, three-bay and n-bay shifts can be considered, where n is an integer. It will be immediately clear that the structure is symmetric for each of these operations, which consist of the basic operation compounded with itself. A loading case that is co-symmetric in relation to the one-bay shift, is also co-symmetric for any n-bay shift. The following statement is perhaps more revealing: a loading case that is contra-symmetric in relation to the one bay-shift, is contra-symmetric for the n-bay shift with n odd, but co-symmetric if n is an even number. Superposition of two loading cases, of which one is co-symmetric and the other contra-symmetric in relation to the one-bay shift, results in a loading case that is co-symmetric for any shift over an even number of bays, first of all for the two-bay shift. Conversely, only a loading case that is (spatially) periodic with a smallest period of two bays, can be separated into loading cases that are co-symmetric and contra-symmetric for the one-bay shift.

For any other loading, separation into simpler cases will be possible only (if at all) when simple cases of a new kind are discovered.

Co- and contra-symmetry for the one bay-shift are particular instances of factor symmetry, a more general concept to be introduced now. The loading case shown in the diagram below, consists of concentrated forces that increase by a factor λ for each consecutive bay. The loading as a whole can be transformed into itself by a one-bay shift together with a multiplication by the factor λ, the order being immaterial. According to the general definition of symmetry one can state that the loading case is symmetric for the transformation as described, which is its symmetry operation. The loading case will be called factor-symmetric.

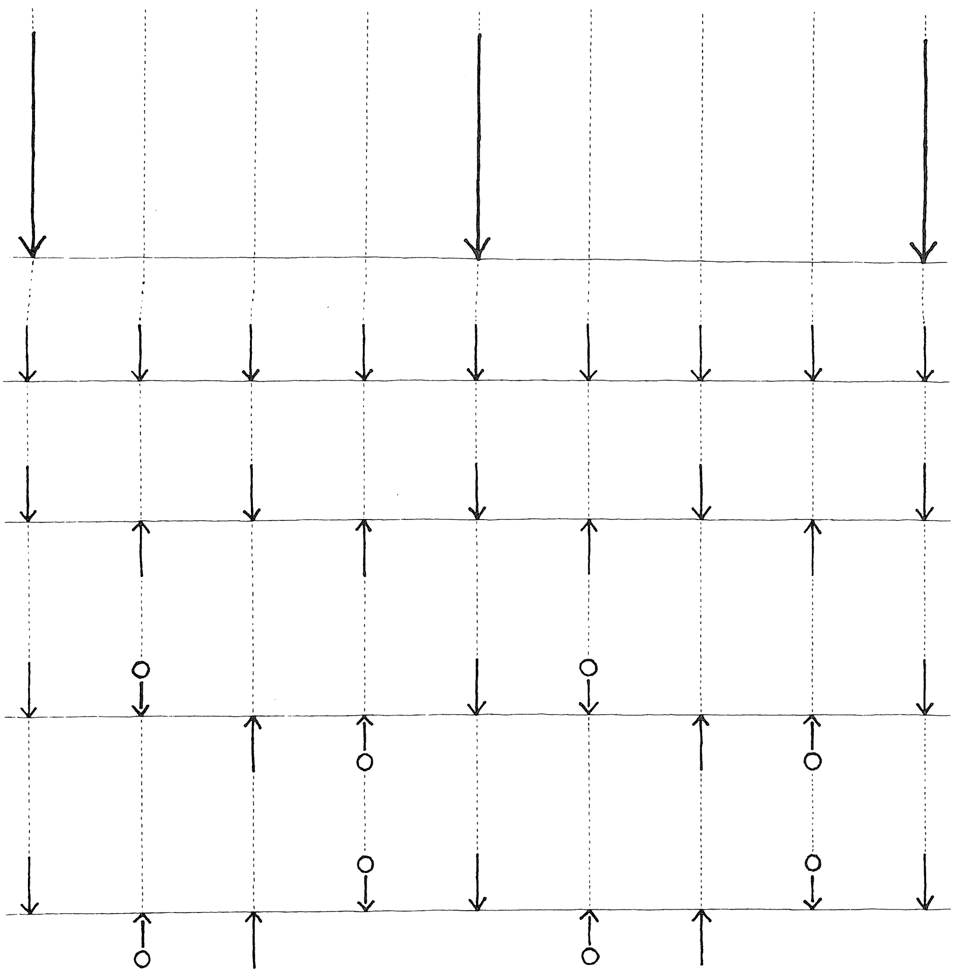
When the factor is larger than 1 the forces have increasing magnitude as one considers bays further to the right, and for a factor smaller than 1 the same happens towards



the left. Doubts could be raised with regard to an earlier assumption, viz. that the phenomena at the far edges have no effect on the results for centrally located bays; since the forces will be quite large for one far edge, it is hard to say beforehand what their influence might be. This difficulty will be ignored for the moment, and in due course possibilities will appear to either circumvent or resolve the problem.

In linear mechanics a deformation pattern with the same factor symmetry as the loading will satisfy the equations of the displacement method for the consecutive joints of the structure. It is possible to consider the displacements of the joints in a single bay as the basic unknowns,since the displacements of adjoining bays can be expressed in these unknowns. Again the equations for six unknowns are sufficient to find the results, so the general case with factor symmetry is no more difficult than its special instances of co- and contra-symmetry. These appear as such when the factor λ is taken to be 1 and -1 respectively.

A factor-symmetric loading is also transformed into itself by a two-bay shift and multiplication by λ squared, i.e. the basic symmetry operation applied twice. When λ squared happens to equal unity, the two-bay shift alone will do. By taking both the square roots (1 and -1) of unity, the earlier result is recovered, that loading cases which are either co- or contra-symmetric for the one-bay shift, are both co-symmetric for a two-bay shift. The result has been linked to the profitable use of symmetry considerations, and the separation into simple cases of a slightly more complicated case that is periodic with a smallest period of two bays.

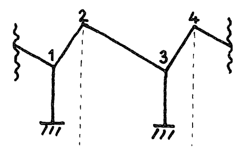


The result can be generalized by considering all factor-symmetric cases, where a power of λ equals unity. Complex roots will not be avoided; they are in fact essential for a widening of the possibilities. In the next example use is made of all the fourth-power roots (1, − 1, i and − i) of unity, in order to obtain the full set of simple cases that transform into themselves when a four-bay shift is applied. Any loading case that is periodic with a period of four bays, can be viewed as the result of superposition, where the four simple cases are combined in appropriate ratio. The loading case of the example consists of concentrated forces on every fourth bay, the bays in between being free of loading. In the diagram the separation into simple cases is also shown. A special kind of arrow is used as the representation of loads that have an imaginairy value. When one arrives at this result by easy stages, it does not look particularly difficult; when presented as a puzzle to a few experts in structural analysis however, it turned out to be altogether intractable. A first step of separation into parts exhibiting co- and contra-symmetry for the two-bay shift is easy enough, but the further separation of the last part presents an apparently insurmountable difficulty.

As mentioned before, the simple cases with factor symmetry can be solved by for instance the displacement method, which leads to equations with six unknowns. Although the preceding example requires the use of complex numbers, this makes the calculation only a little more difficult. It is perhaps time to replace the earlier hand-waving explanation of this work by a more formal treatment.

First one has to number the joints. The connections at the foundation do not require a number, since their displacements are zero. Somewhere in the central part of the structure, one will arrive at high numbers like 81, 82 and so on for the joints whose displacements are free. However, in order to make the equations more readable, one-digit numbers will be used exclusively (perhaps the preceding joints are numbered by zero and negative numbers... no matter, they will not be looked at again). Next a partitioning of the stiffness matrix into submatrices of order three is carried through, corresponding to the degrees of freedom at each of the joints. The displacement vector and the force vector are similarly subdivided and all partitions are labeled by subscripts that refer to joint numbers. The letters S, v and k will be used as kernels for the stiffness matrix, displacement vector, and force vector respectively; also of course for their partitions. The translational components and the rotation of joint 1 for instance, are contained in partition $\underline{v}_1$ of the displacement vector.

The equations of the displacement method are derived from the equations of equilibrium for the joints. As a basis for further discussion, the equations for joints 2 and 3 will serve. They take the following form in compact notation:



$$S_{21}\underline{v}_1 + S_{22}\underline{v}_2 + S_{23}\underline{v}_3 = \underline{k}_2$$

$$S_{32}\underline{v}_2 + S_{33}\underline{v}_3 + S_{34}\underline{v}_4 = \underline{k}_3$$

Since the one-bay shift that transforms the structure into itself, also takes every joint to a similar joint, whose number differs by two from the original, the submatrices of the stiffness matrix are identical to submatrices whose index and column number differ by plus or minus two. The partitions of the displacement vector and the force vector exhibit factor symmetry; they increase by the factor λ in consecutive pairs.

$$S_{33} = S_{11}\,; \; S_{34} = S_{12}$$

$$\underline{v}_3 = \lambda \underline{v}_1\,; \; \underline{v}_4 = \lambda \underline{v}_2$$

$$\underline{k}_3 = \lambda \underline{k}_1\,; \; \underline{k}_4 = \lambda \underline{k}_2$$

The equations are modified accordingly, and also interchanged.

$$\lambda S_{11} \underline{v}_1 + (S_{32} + \lambda S_{12}) \underline{v}_2 = \lambda \underline{k}_1$$

$$(S_{21} + \lambda S_{23}) \underline{v}_1 + S_{22} \underline{v}_2 = \underline{k}_2$$

In the equations, which are still expressed in compact matrix notation, each line represents three equations when written in full, because the submatrices used are of order three. They are routinely found according to the displacement method. For a given value (possibly complex) of the factor λ the calculation will proceed with one of the known procedures for the solution of linear algebraic equations. Account must be taken of the possibility that the coefficients may be complex. Furthermore, the matrix of the equation system is asymmetric, except in the case $\lambda = 1$, also known as the case of co-symmetry for the one-bay shift.

There is one other question, that has not been addressed earlier, but is somewhat thought-provoking. What can be said about deformation patterns, and the accompanying internal stress distributions, in cases where the structure carries no load except at the far edges? The answer again uses the assumption of factor symmetry with a factor unknown at present. The problem will be described by the equation system as derived before, but in this case the right-hand sides are zero, and the factor λ has to be chosen in such a way that the equations have a non-trivial solution, one different from zero that is to say. This belongs to a class of problems known as eigenvalue problems, in fact even to one of the more difficult subclasses. Terms with the factor λ occur throughout the matrix; neither they nor the terms without λ are symmetrically distributed. The required value of λ (called the eigenvalue) will in general be complex.

The difficulties notwithstanding, one might profitably invest the effort required to solve this problem. In analogy with differential equations, one tries to obtain the solution to the homogeneous system, in order to go beyond particular solutions to general solutions, taking account of any boundary conditions that may occur. No longer does one require the assumption that the extent of the structure is infinite or sufficiently large to treat is as such. Any structure that repeats a basic form many times,can be dealt with regardless of its length and the effects that phenomena at the far edge may have. In fact the solutions now obtained represent how such phenomena are propagated through the structure.

In continued analogy with differential equations, the simple cases with factor symmetry are related to substitutions of goniometric functions, or rather their complex form derived from the Euler-de Moivre formula. The link with complex roots of 1 makes that obvious.

References

1. Coxeter, H. M. S., Introduction to Geometry, John Wiley and Sons, 1961.
2. Livesley, R. K., Matrix Methods, in: Hyperstatic Structures, chapter contributed to book by J. A. L. Matheson, Butterworths, 1959.
3. Wilkinson, J. H., The Algebraic Eigenvalue Problem, Clarendon, 1965.
4. Blake, W., The Tyger, in: Songs of Experience, author's edition, 1794.