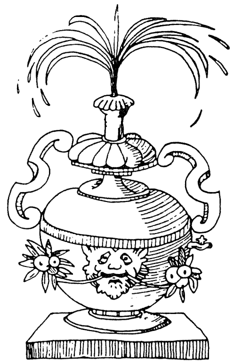
Heron's Fountain



A feature appearing from time to time, when the occasion allows the presentation of ideas with surprising elements, which may have something in common with the playful inventions of Heron of Alexandria, after whom this journal is named.

What shoulder, and what art

by I. M. MORTELHAND

The buckling problem has been studied since structural analysis was in its infancy [1]. Petrus van Musschenbroek published experimental results in 1729 which tended to show that the critical load for a compressed column is inversely proportional to the square of the column length. Work by Leonard Euler in 1744 and 1759 provided a theoretical basis in the form of an expression for what is now generally known as the Euler load:

$$F_E = \frac{\pi^2 EI}{l^2} \tag{1}$$

The notation EI was not used at first, since the bending stiffness was considered as a single quantity. Later contributions have clarified how it can be factorized into the modulus of elasticity and the purely geometrical quantity I which depends on the shape of the cross-section. On the other hand it was soon pointed out that linear-elastic behaviour, as implied by the use of E, is hardly certain for materials used in construction. If the material involved passes through a linear-elastic stage for low stresses, there still remains a range of validity, albeit restricted, for Euler's formula. In this context it is usually cast in another, more appropriate form:

$$\sigma_E = \frac{\pi^2 E}{\lambda^2} \quad \text{where} \quad \sigma_E = \frac{F_E}{A}; \quad \lambda = l\sqrt{\frac{A}{I}} \tag{2}$$

Buckling occurs at a low stress level for high values of λ, the slenderness ratio. The emergence of this insight prompted analysts to make the now familiar distinction between slender columns, for which Euler's formula is considered to be valid, and short columns, for which non-linear material behaviour comes into play. Hence attempts were made to establish buckling curves for several types of material. A buckling curve is a diagram that shows the dependence of the critical stress on the slenderness ratio. Its

general appearance is somewhat like a shoulder (see Fig. 1); this fact and a source of poetical inspiration [2] provided the title of the present article.
Its focus of attention will be one detail of the curve, the shoulder joint so to speak: the transition between the parts that refer to short and slender columns respectively. First however, the brief historical review is resumed.
Many proposals for buckling curves were based on a number of tests, augmented by some artfulness on the part of the researcher. If he thought a smooth curve for all columns was the most reasonable form, he would draw the part for short columns in such a way that it arrived tangentially at the junction with the Euler curve. Others thought differently and drew an angular transition.
From the theoretical side the concept of the tangent modulus was introduced to provide a rational basis for tracing the curve. According to this theory, the answer as to whether the junction is smooth or angular could be derived from precise knowledge about the stress-strain relationship of the material involved. As long, however, as materials with erratic behaviour were used, the matter was to remain in doubt.
Developments in foundry practice during the latter half of the 19th century (Bessemer-Thomas and Siemens-Martin processes) provided more control over the properties of materials derived from iron ore. The demands of the structural profession were met by ductile steel. It yields at a certain stress level, a property which is deemed desirable because the yield stress is maintained throughout fairly large deformations. Simple yet trustworthy calculation methods for ductile steel structures (simple because only the final plastic stage was taken into account) met with much success from the beginning of the 20th century onwards.
Another remarkable property of ductile steel is that the linear-elastic stage seems to continue until the yield stress is reached. Its the tangent modulus theory could be relied on, the buckling curve would also be of the simplest kind imaginable. It would consist only of a straight line and the Euler curve, meeting at an angle. The straight line would be horizontal, because the yield stress would be the limit for short columns.

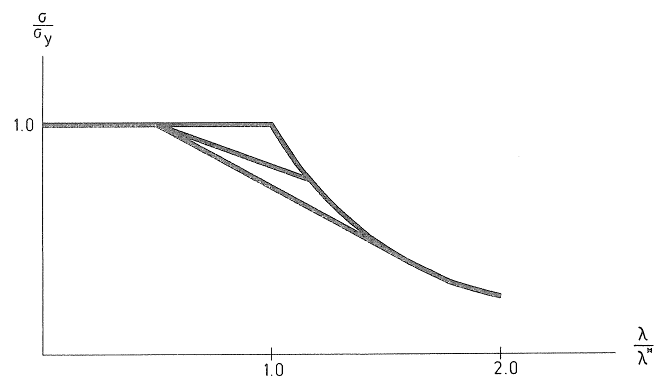


Fig. 1. Various types of shoulders.

$$\begin{array}{ll} \sigma \leqq \sigma_y & \text{for} \quad \lambda \leqq \lambda^* \\ \sigma \leqq \sigma_E & \text{for} \quad \lambda \geqq \lambda^* \end{array} \quad \lambda^* = \pi \sqrt{\frac{E}{\sigma_y}} \tag{3}$$

Experimental results do not support this simple view at all; especially in the neighbourhood of the transition the observed buckling stress is appreciably lower than would be expected. The tangent modulus theory appears not to hold even for the simplest of materials, so one wonders how much credence it deserves in more complicated situations. Still, textbooks continue to treat the tangent modulus theory with deference, possibly because in the late 19th and early 20th century it was developed by persons justly renowned for other achievements.

In contrast a much more realistic approach to the buckling problem, also dating from the 19th century, long dwelt in undeserved obscurity. Even now several sources must be consulted in order to trace the development of the idea that imperfections should be taken into account. In [1] due credit is given to the very early contribution (1807) by Thomas Young. He considered both the case of a column with a slight initial curvature and the case of a straight column with an eccentric force in the direction of the axis, and gave the correct solution for each. The possible application to design did not receive attention for half a century. The first proposal of this kind was published in 1858 by Scheffler. Its description in [1] is unfortunately brief; even the title of the publication is rendered incompletely. Another article, published in 1886 by Ayrton and Perry [3], is not even mentioned. In fact only the name of the second author was remembered when in 1925 the thread was picked up again by Robertson [4]. For some reason or other Ayrton was left to starve on an uninhabited island.

For ease of calculation authors generally took a sine curve as the form of the axis; other cases can be dealt with approximately by taking the first term of a Fourier series as a representation of the true form. The case of eccentric loading with equal eccentricity at both ends can also be treated in this way, the amplitude of the representative sine curve being taken as $4/\pi$ times the eccentricity.

Although the names of the pioneers are almost forgotten, their formula is now very well known. Derived by applying an amplification factor to the bending moment in the midsection, then combining the average compressive stress σ with the bending stress to determine when yielding begins in the farthest outlying fibre, it takes the following form after some algebraic manipulation:

$$(\sigma_y - \sigma)(\sigma_E - \sigma) = \frac{Aw_0}{Z} \sigma \sigma_E \tag{4}$$

Here w_0 denotes the amplitude of the sine curve that describes the form of the column axis and Z is the section modulus; other symbols have been used earlier. The values of A, Z and σ_y are known for a given cross-section and material; furthermore σ_E can be derived from E and the slenderness ratio, which is the independent variable of the buckling curve. The imperfection amplitude w_0 is estimated with some support of measurements, but it remains the most uncertain of the quantities. In the ideal case of the perfectly straight column the right-hand side of the formula becomes zero, leading us back to the diagram consisting of a straight line and the Euler curve. This is now considered as an upper bound, the ideal as opposed to reality beset with imperfection. Taking a value of the amplitude w_0 different from zero, we can still easily solve (4) as a quadratic equation for the average compressive stress. The lowest of the two roots is the desired result. It serves as the dependent variable when the whole range of the slenderness ratio is explored; of course the imperfection amplitude may also vary across this range. In view of the earlier considerations it is remarkable that just one equation gives the whole of the curve: there is no natural separation between parts that describe short and slender columns. By way of illustration, in Fig. 2 the diagram has been drawn for Aw_0/Z equal to $0.1\lambda/\lambda^*$, which corresponds roughly to $w_0 = l/2000$, a rather small imperfection. The diagram is actually nearer to the upper bound than a realistic buckling curve should be, but it brings out a salient point. The influence of the imperfection is obviously much larger at the transition than for either low or high values of the slenderness ratio. This observation, combined with the assumption that Aw_0/Z is a small number, may now be exploited in order to derive useful approximations for the various cases.

For fairly short columns σ and σ_y are nearly equal and both are small when compared with σ_E; this allows the following approximations of equation (4) and its solution:

$$\sigma_y - \sigma = \frac{Aw_0}{Z}\sigma_y \quad \text{and} \quad \sigma = \sigma_y\left(1 - \frac{Aw_0}{Z}\right) \tag{4a}$$

Conversely, for fairly slender columns σ and σ_E are nearly equal and both are small when compared with σ_y; this time different approximations of equation (1) and its solution are possible:

$$\sigma_y(\sigma_E - \sigma) = \frac{Aw_0}{Z}\sigma_E^2 \quad \text{and} \quad \sigma = \sigma_E\left(1 - \frac{Aw_0}{Z}\cdot\frac{\sigma_E}{\sigma_y}\right) \tag{4b}$$

Comparison of (4a) and (4b) shows that the relative shortfall (difference of the buckling stress with the upper bound, divided by the bound itself) is directly proportional to the

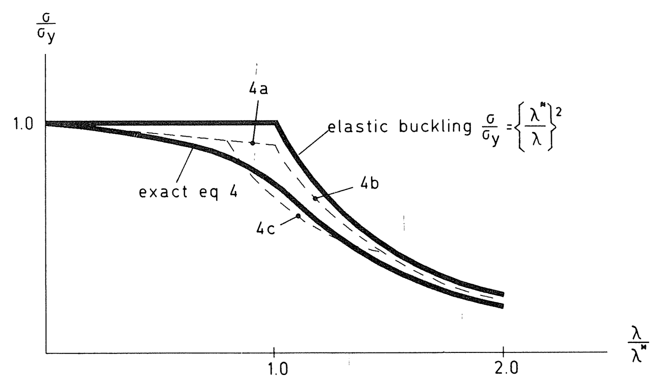


Fig. 2. The influence of the imperfection $\frac{Aw_0}{Z} = 0.1\,\frac{\lambda}{\lambda^*}$

imperfection amplitude in either case. But whereas in (4a) this is the sole factor, in (4b) the relative shortfall is also proportional to σ_E and therefore inversely proportional to the square of the slenderness ratio. By a fortuitous coincidence (4a) and (4b) give the same result at the transition where $\sigma_E = \sigma_y$, setting a trap for the unwary (see Fig. 2). One might be tempted to make the naive assumption that (4a) and (4b) together constitute a good approximation to the buckling curve across the whole range of the slenderness ratio. The truth is rather different.

Some care must be exercised in seeking a reliable approximation for the intermediate case. First the left-hand side of (4) is rewritten as $\{\sigma - \frac{1}{2}(\sigma_y + \sigma_E)\}^2 - \frac{1}{4}(\sigma_y - \sigma_E)^2$. Since it provides the definition of λ^* the relation $\sigma_E = \sigma_y$ holds exactly at the transition and approximately in its neighbourhood, and we may neglect the second term. For the right-hand side of equation (4) we use the approximation that $\sigma_E \sigma \approx \frac{1}{4}(\sigma_y + \sigma_E)^2$. This can be justified by the fact that the right hand term is small. The result of these manipulations is:

$$\{\sigma - \tfrac{1}{2}(\sigma_y + \sigma_E)\}^2 = \frac{Aw_0}{Z}\,\tfrac{1}{4}(\sigma_y + \sigma_E)^2 \quad \text{or} \quad \sigma = \frac{\sigma_y + \sigma_E}{2}\left(1 - \sqrt{\frac{Aw_0}{Z}}\right) \tag{4c}$$

On substituting $Aw_0/Z = 0.1\lambda/\lambda^*$, as before, we arrive at:

$$\lambda \ll \lambda^* \qquad \sigma = \sigma_y(1 - 0.1\lambda/\lambda^*) \tag{5a}$$

$$\lambda \approx \lambda^* \qquad \sigma = \frac{\sigma_y + \sigma_E}{2}\,(1 - \sqrt{0.1\lambda/\lambda^*}) \tag{5b}$$

$$\lambda \gg \lambda^* \qquad \sigma = \sigma_E(1 - 0.1\lambda^*/\lambda) \tag{5c}$$

The occurrence of a square root in (5b) explains how a small imperfection can have a disproportionate influence in the transition zone. For example, with $Aw_0/Z = 0.1$ at $\lambda = \lambda^*$ the approximation gives 0.316 for the relative shortfall of σ/σ_y (exact 0.27, see Fig. 2).

A peak in the shortfall is to be expected, since it bridges the gap between the smooth buckling curve and the upper bound diagram consisting of a curve and a straight line meeting at an angle. Even if the imperfection would be imperceptibly small, the buckling curve would still have a slightly rounded corner.

About two decades ago the ECCS (European Convention for Constructional Steelwork) carried out an extensive testing program on columns of various types with much theoretical work in its wake [5], that paved the way for a rational approach to the buckling problem in European codes. With the purpose of obtaining realistic buckling values, the columns were tested in the form they would have in constructional work; no attempt was made to remove imperfections. The accompanying theoretical studies gave attention to geometrical imperfections, residual stresses and redistribution of stresses in the elastoplastic stage. These aspects will not be gone into here, except to note that redistribution has a favourable effect mainly for short columns, that residual stresses depend strongly on the shape of the cross-section and the method of fabrication (their influence being generally unfavourable, but sometimes slightly favourable), and finally that geometrical imperfection increases with the column length, an effect that is unfavourable for slender columns. For section types with the least influence of residual stresses, a buckling value of about $0.7\sigma_y$ is found at the transition (as in the example of the present article), but lower values of $0.5\sigma_y$ or $0.4\sigma_y$ are possible.

A recent trend in the formulation of codes is to make some kind of reference, however indirectly, to probabilistic methods. Since imperfections are never intentional, their influence on the buckling strength of columns is certainly a stochastic phenomenon. Analysis of this aspect for the test results just mentioned has brought out a complicated picture. The example of the present article is continued as an academic exercise, in order to draw attention to another peculiarity of the subject.

Suppose the (geometrical) imperfection amplitude to be a stochastic variable with a Gaussian probability distribution and let its mean value be zero. Negative values of the amplitude are not substituted as such in the Young-Scheffler-Ayrton-Perry-Robertson formula. The bending stress is compressive in the fibres at the inner side of the bend; its extreme value at this side is to be combined with the average compressive stress. It comes to the same thing when one always uses $|w_0|$, the absolute value of the amplitude. The probability distribution is then folded over to the positive side, giving the following result in terms of a normalized variable (see Fig. 3):

$$\Phi(u) = \sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}u^2} \quad \text{with} \quad u \geqq 0 \tag{6}$$

A probability distribution of type (6), such as we have now assumed for the absolute value of the amplitude, would also very nearly describe the related distribution for the shortfall in the buckling strength of fairly short and fairly slender columns. The approximations (4a) and (4b) for these cases show the shortfall to be nearly proportional to the amplitude, hence the similarity of the distribution type. This would entail a value zero for the modal value, i.e. the value with the highest probability density, of the shortfall. The buckling strength itself would have its modal value at the upper bound; in experiments many columns would buckle under loads quite near to the theoretically

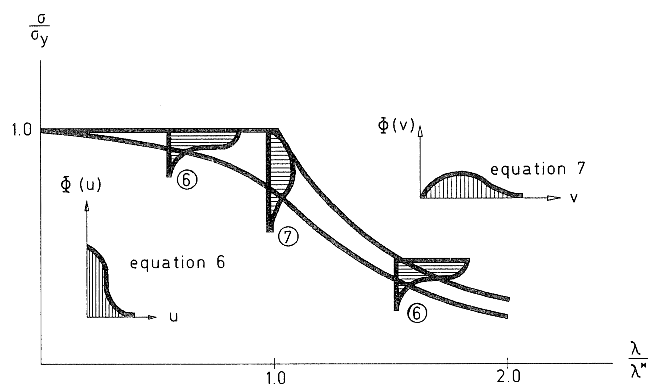


Fig. 3. The probability density of the critical stress is zero at the shoulder tip.

attainable maximum. Remember, this is an academic exercise: for real test results the effect is obscured beyond recognition by other factors coming into play, e.g. the yield stress itself is a stochastic variable.

Approximation (4c) shows the shortfall at the transition to be proportional to the square root of the amplitude. The relevant probability distribution type is obtained when a transformation $v=\sqrt{u}$ is carried out. Since this relation is monotonous (v increases with u everywhere), the probabilities associated with an interval du and the transformed interval dv are the same:

$$\Phi(v)\ \mathrm{d}v=\Phi(u)\ \mathrm{d}u \quad \text{and hence} \quad \Phi(v)=2v\sqrt{\frac{2}{\pi}}\,e^{-\frac{1}{2}v^4} \tag{7}$$

This result typifies the influence of the geometrical imperfection at the transition value of the slenderness ratio. Comparison of diagrams for the distribution types (6) and (7) shows a large difference: the latter is much more like a bell curve. The probability density drops to zero at the boundary; buckling test results will clearly fall short of the maximum attainable load (see Fig. 3).

Summarizing, the present article endeavours to draw attention to the following points. When one considers the influence of a small geometrical imperfection on the buckling strength of ductile-steel elements, the shortfall will be proportionally small for fairly short and slender columns. At the transition between short and slender columns the shortfall is much more pronounced, because there it is proportional to the square root of the imperfection. This result may be combined with probability considerations, restricted here to an academic exercise in which only the geometrical imperfection is treated as a stochastic variable. The occurrence of the square root is reflected in the probability distribution for the buckling strength at the transition, of which even the type differs markedly from that in other places.

References

1. TIMOSHENKO, S. P., History of the Strength of Materials, McGraw-Hill 1953.
2. BLAKE, W., The Tyger, Author's edition 1794.
3. AYRTON, W. E. and PERRY, J., On Struts, The Engineer, 1886.
4. ROBERTSON, A., The Strength of Struts, Inst. Civ. Eng., 1925.
5. Several authors/articles, Construction Metallique, 1970.