# Identification of elastic properties by a numerical-experimental method

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**Abstract**

For the determination of material parameters it is common practice to use specimens with well defined geometries. The design of the samples and the choice of the applied load are meant to lead to a simple, often homogeneous, stress and strain distribution in a part of the sample. However, application to composite systems raises a number of problems.

In this paper a different approach is presented based on the combination of three elements:

a. The use of digital image analyses for the measurement of inhomogeneous strain distributions on multi-axially loaded objects.
b. Finite element modeling.
c. Application of systems identification.

The method is tested by means of experiments on an orthotropic elastic membrane. The results are compared with classical testing results.

## 1 Introduction

The present paper deals with the development of a method for the experimental characterization of solids with complex properties. Common features of the solids are anisotropical behavior and inhomogeneous properties. Examples are nearly all biological tissues but can also be found in technical materials like injection moulded products with short fibers and long fiber composites. The method is aimed at an experimental quantitative determination of material parameters in constitutive equations. It is assumed that preliminary research yielded a fairly good idea of what type of constitutive equation is suitable for describing the behavior of the material under consideration. In traditional testing it is common practice to use specimens with well defined geometries. These samples are loaded in testing machines under well controlled testing conditions. The design of the samples and the choice of the applied load are meant to lead to a simple, often homogeneous, stress and strain distribution in a part of the sample. Examples of these kinds of tests are bars in tension and circular rods in torsion. It is clear that the procedure, when applied to composite systems, raises a number of problems:

a. Composites may have inhomogeneous properties which makes it impossible to obtain a homogeneous stress and strain distribution in the sample.

b. It is not always possible to make samples of a determined sample shape for fiber-structure reinforced composites without causing deterioration in the internal coherence of the structure.
c. The number of experiments needed for an adequate characterization of the material is large.

In the approach presented in this paper, more freedom for the experiments is created. This approach, which will be referred to as "the identification approach", offers new possibilities for the characterization of complex materials. The restrictions of the homogeneous stress and strain distribution are relaxed and it is acknowledged that the inhomogeneous strain distribution has to be measured and that the field equations are solved numerically. A consequence of this strategy is that it is difficult to isolate the influence of single parameters. The material properties are not measured directly as in traditional testing. Instead parameter estimation techniques are used to determine the set of unknown parameters, using the complete set of experimental data.

Parameter estimation is a main issue in system identification (see e.g. Norton [1]). Applications of system identification can be found in a variety of disciplines. In structural engineering, system identification has been applied primarily to dynamic systems, with the purpose of identifying parameters which hardly lend themselves to direct measuring, such as damping characteristics (Natke [2]). Applications in continuum mechanics are rare [3–11]. In general simple constitutive relations are considered, where structural modeling errors are denied by using artificially generated data.

In the present paper the identification approach is verified by experimental results. In the next section an identification experiment will be described. The numerical model of this experiment will be given in section three. The identification theory will be quoted shortly in section four, whilst in section five the results of the identification will be presented. Results of traditional experiments with the same material will be given in section six. Here the comparison of the identification approach and the traditional testing will be discussed.

## 2 Identification experiment

The objects used in the experiment are allowed to have an arbitrary shape. The applied load is chosen in such a way that the object shows an inhomogeneous strain distribution. The underlying assumption of the method is that the strain distribution together with the applied load contain enough information to determine the material parameters. It is obvious that in general a homogeneous strain distribution is inconsistent with this assumption.

Fig. 1 is a schematic drawing of the experimental setup. This experiment will be used as a test for the identification approach. A membrane of woven and calendered textile was clamped along one edge and was free to deform at the other edges. The membrane ($100 \times 100$ mm$^2$) was loaded with two forces ($F_1 = 0.1$ kN and $F_2 = 0.05$ kN). The deformations were kept small (up to 3%). The material behaves in an anisotropic way and has homogeneous properties. The sample is extracted in such a way that one

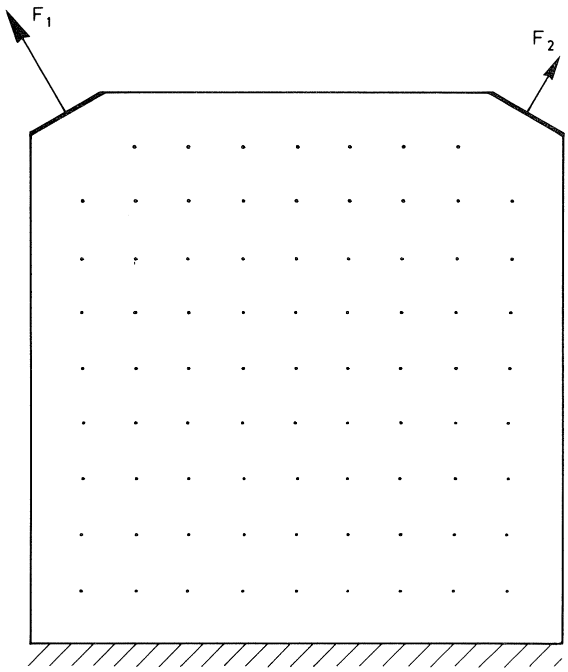


Fig. 1. Experimental setup.

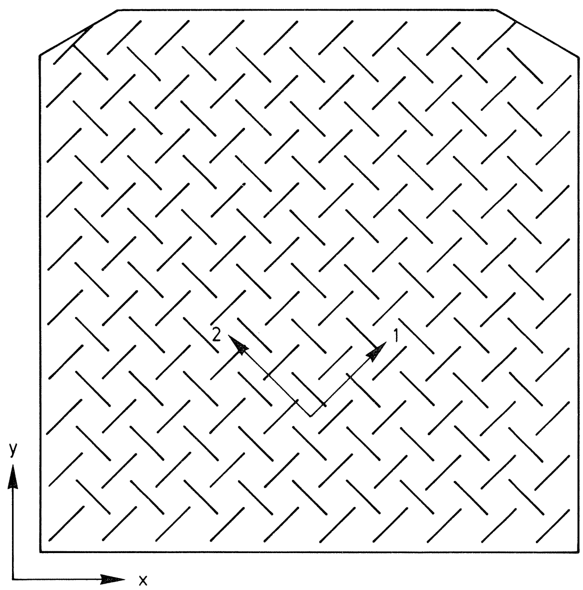


Fig. 2. Positive rotation of principal material axes from arbitrary *xy*-axes.

principal material axis has a positive rotation of 45° from the clamped edge (Fig. 2). The strain distribution on the surface of the object is measured by mounting retro reflective marks ( ⌀ 1 mm) on the object surface. The positions of these marks are measured by means of a video tracking system [12] based on random access cameras. Illumination from the camera position causes reflections from the marks with higher intensity than the environment, which can be used to identify the marks. After a search scan the tracking system identifies the marks and defines a window around each mark. In window scan mode only these windows are scanned, and in each sample period the centroid of the window position is adjusted accordingly.
The geometry of the sample is measured by putting additional marks on the edges of the surface. Moreover marks are attached to the strings inducing the two forces, in order to measure the directions of the forces.

## 3 Numerical model

The experiment of the previous section is modeled by means of the finite element method with 4-noded, isoparametric, plane stress elements (Fig. 3). The material behavior is assumed to be orthotropic, linear elastic. Moreover the material is assumed to have homogeneous properties. The quantitative behavior can be described with 5 (engineering) parameters: two Young's moduli ($E_1$ and $E_2$), a Poisson's ratio ($\nu_{12}$), the shear modulus ($G_{12}$) and a parameter which indicates the positive rotation of the mate-

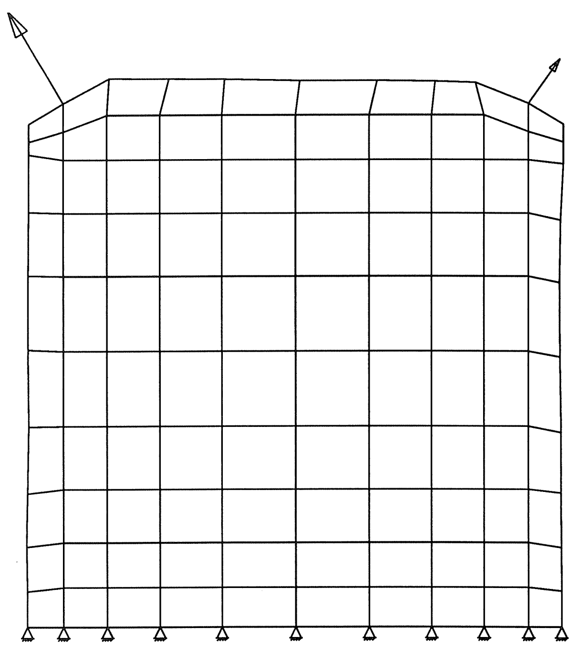

Fig. 3. Finite element model.

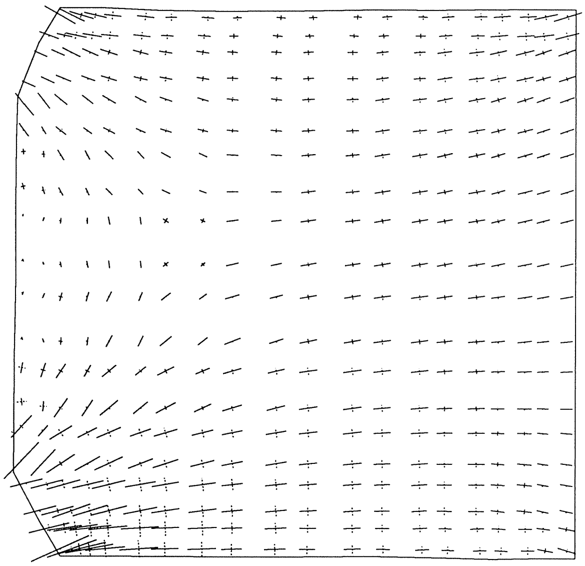

Fig. 4. Principal strain distribution.

rial axis from the arbitrary model axis (tangent $(\alpha)$, where $\alpha$ denotes the rotation). For the finite element calculations DIANA [13] is used. Fig. 4 shows a typical modeled strain distribution.

## 4 Identification method

In this section an outline of the identification method used is described. The method is based on the sequential minimum variance approach and resembles Kalman filtering techniques. The observational data are assumed to consist of a set of columns with data $\{y_k\}$, $k = 1, ..., N$. The observational data of the experiment described in this paper are collected in a single column $\boldsymbol{y}_1$. Column $\boldsymbol{y}_1$ contains 158 displacement components of the marks. These displacement components are modeled with the help of the finite element model of the previous section. The modeled displacements are considered to be a nonlinear function of the material parameters:

$$\boldsymbol{y}_1 = \boldsymbol{h}_1(\boldsymbol{x}) + \boldsymbol{v}_1 \tag{1}$$

where $\boldsymbol{x} = (E_1, E_2, \nu_{12}, G_{12}, \tan(\alpha))^{\mathrm{T}}$ is a column with material parameters, $\boldsymbol{h}_1$ is a finite element model for the measured displacements $\boldsymbol{y}_1$, and $\boldsymbol{v}_1$ is a column of observation errors.

The basic estimation problem is the use of the observed variables $\boldsymbol{y}_1$ to estimate parameter column $\boldsymbol{x}$. The estimator can be specified from the model (1), an uncertainty model for $\boldsymbol{v}_1$ and a priori knowledge of $\boldsymbol{x}$. The optimal parameter column minimizes the following quadratic expression:

$$S_1 = (\boldsymbol{y}_1 - \boldsymbol{h}_1(\boldsymbol{x}))^{\mathrm{T}}\boldsymbol{R}^{-1}(\boldsymbol{y}_1 - \boldsymbol{h}_1(\boldsymbol{x})) + (\hat{\boldsymbol{x}}_0 - \boldsymbol{x})^{\mathrm{T}}\boldsymbol{P}_0^{-1}(\hat{\boldsymbol{x}}_0 - \boldsymbol{x}) \tag{2}$$

where $\hat{x}_0$ is an initial guess for the parameter column $x$. In weighted least squares estimation the matrices $R$ and $P_0$ are chosen on the basis of engineering judgement. The least squares estimate does not make any use of the statistics of the observation errors. In many applications, it is not uncommon for the mean and variance of the observation error to be known. Minimum variance estimates utilize this extra information, which results in specific choices for $R$ and $P_0$. In minimum variance estimation $R$ represents the covariance matrix of the observation error $v_1$. Matrix $P_0$ represents the covariance matrix of the estimation error in $\hat{x}_0$. Generally: the large the $P_0$, the smaller the influence of $x_0$.

Solving the non-linear inverse problem, defined by (1) and (2), leads to an iterative scheme, which results in an estimation $\hat{x}_1$ for $x$ and in a covariance matrix of the estimation error $P_1$ (see [14]). In each iteration the program executes $n + 1$ finite element calculations, where $n$ is the number of parameters (in this example equal to 5). The $n$ calculations are carried out to determine a matrix $H_1$ numerically, as a linearization of $h_1$ with respect to the most recent estimation $\hat{x}_0$. The sequential property of the estimator is clear when a column $y_2$ becomes available with new observational data. This may be data from a different experiment on the same material, or in the case of visco-elastic behavior, it may be observations from another point in time. These data can be used together with the initial conditions $\hat{x}_1$ and $P_1$ resulting in an improved estimation $\hat{x}_2$ and $P_2$. In the example of this paper it will be shown that the data contained in $y_1$ is sufficient for the characterization of the material behavior.

The above estimator is implemented as an extra module PAREST in the finite element code DIANA.

## 5 Identification results

To initiate the recursive identification method, an initial guess $\hat{x}_0$ for the parameter values and an initial guess for the error covariance of $\hat{x}_0$, matrix $P_0$, are needed. The following value is chosen $\hat{x}_0^T = \{0.70, 0.30, 0.30, 0.10, 1.50\}$ with dimension [kN/mm$^2$, kN/mm$^2$, -, kN/mm$^2$, -]. We consider $P_0$ to be diagonal with $10^{-2}$ for all diagonal elements, corresponding with the expectation of the squared errors in the initial guess. The accuracy of the measured displacements can be expressed by setting the covariance $R$. We consider that the observation errors are mutually independent, which means that matrix $R$ is diagonal. The diagonal elements are set to $10^{-4}$ [mm$^2$].

Fig. 5 shows estimations of the five material parameters as a function of the iteration counter, starting with the guess $\hat{x}_0$. As can be seen, the estimations converge. The resulting parameter column is $\hat{x}_1^T = \{0.56, 0.57, 0.22, 0.08, 1.05\}$. Fig. 6 shows the parameter estimation with a completely different initial guess $\hat{x}_0^T = \{2.00, 4.00, 0.25, 0.50, 1.00\}$. It can be seen that the parameter estimator is robust and that the parameters converge to the same values.

A first indication of the reliability of the estimated parameters is a comparison between experimental and modeled displacements, using the estimated parameters in the latter. The filled circles in Fig. 7 represent the measured positions of the marks at the time

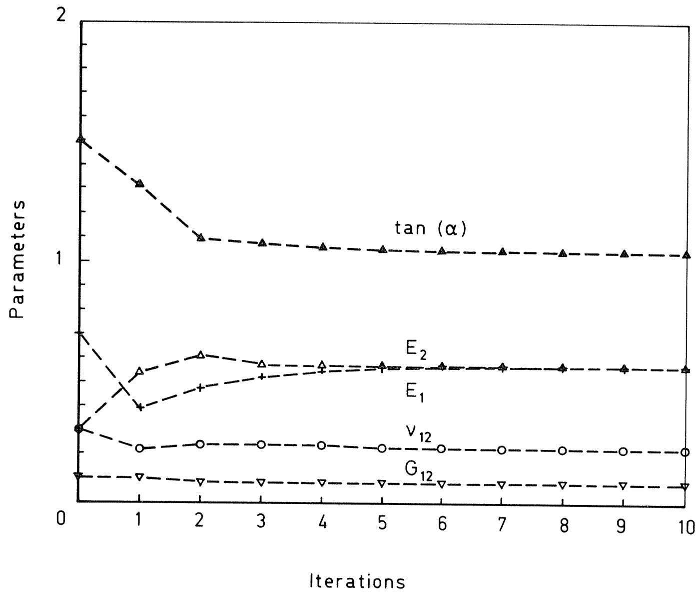


Fig. 5. Parameter estimation.

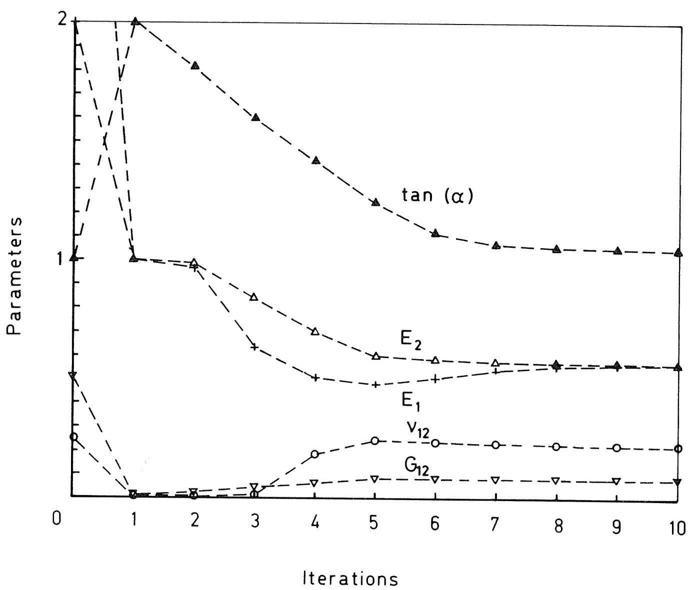


Fig. 6. Parameter estimation based on a completely different initial guess.

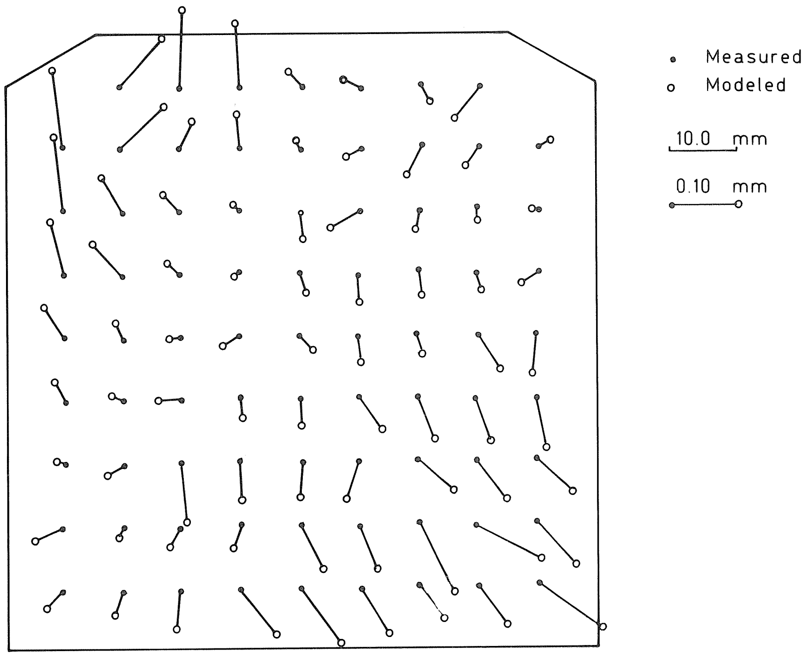


Fig. 7. Residuals: modeled and measured positions of the marks.

when the sample is loaded. The open circles in the picture represent the modeled positions of the marks. In order to obtain useful information from the picture, the distances between the open and filled circles are multiplied by a factor of 100! In general the residuals appear to be small. Another indication of the reliability of the estimated parameters is the variance of the estimation errors. The diagonal elements of $\boldsymbol{P}_1$, belonging to the estimation $\hat{\boldsymbol{x}}_1$, are $\{74, 19, 42, 0.2, 50\} \cdot 10^{-6}$. The square roots of the diagonal elements give an indication of the estimation errors for $\hat{\boldsymbol{x}}_1(1)$ up to $\hat{\boldsymbol{x}}_1(5)$.
Summarizing the results, it can be observed that the identification method is successful and robust. In the next section the results of the identification approach are compared with results of traditional testing.

## 6 Traditional testing results and conclusion

Three uniaxial tension tests are performed on flat pieces of the same material as in section 2. The first two tests are conducted in material 1-direction and 2-direction to determine $E_1$, $\nu_{12}$ and $E_2$. In the third case the loading is at 45° to the material 1-direction. Nine marks are attached to the samples. The displacements are measured with the Hentschel system in order to determine the strains.
The results of the identification approach and traditional testing are summarized in table 1. There is a good agreement between the results of the two approaches. For $E_1$

and $E_2$ the deviation between the results is rather large. A possible explanation is the nonlinear behavior of the material. Another explanation may be the partial destruction of the structure in traditional testing.
The structure of the woven textile is such that two equal Young's moduli would be expected. The symmetry axes of the material are determined optically. It is difficult to specify an accuracy for this traditional determination of parameter $\tan(\alpha)$.
In the identification approach the material direction of the orthotropic material is estimated, together with the other four engineering parameters, from one experiment. The illustrative example described in the present paper shows that this is indeed possible. This property of the identification approach opens possibilities for the characterizing of inhomogeneous products (see [14]). The principal feature is that the method can be used for complex material behavior (e.g. nonlinearity and viscoelasticity).

Table 1: Comparison with traditional testing

| parameter | unit | traditional testing | | identification approach | |
|---|---|---|---|---|---|
| $E_1$ | [kN/mm$^2$] | 0.62 | ($\sigma=0.05$) | 0.56 | ($\sigma=0.01$) |
| $E_2$ | [kN/mm$^2$] | 0.52 | ($\sigma=0.06$) | 0.57 | ($\sigma=0.004$) |
| $\nu_{12}$ | [–] | 0.21 | ($\sigma=0.01$) | 0.22 | ($\sigma=0.01$) |
| $G_{12}$ | [kN/mm$^2$] | 0.080 | ($\sigma=0.005$) | 0.080 | ($\sigma=0.0004$) |
| $\tan(\alpha)$ | [–] | 1.0 | ($\sigma=0.1$) | 1.05 | ($\sigma=0.01$) |

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