A mixed element for geometrically and physically nonlinear shell problems

A. BOUT and F. VAN KEULEN
Faculty of Mechanical Engineering and Marine Technology
Department of Engineering Mechanics
Delft University of Technology

Abstract

A triangular element for geometrical and physical nonlinear shell problems is described. The degrees of freedom are 18 displacement components and 6 rotation-like scalars. A key role in the description is played by a flat reference triangle through the vertices of the element. The governing equations are derived by means of the principle of virtual work, using constitutive equations based on a layered model. The performance of the element is illustrated by two sample problems.

Introduction

Based on Koiter's theory for thin elastic shells [10], a triangular finite shell element has been described in [2, 8, 9]. The element has been applied to geometrically linear and initial post buckling problems of plate and shell structures [9]. In [4] an extension to arbitrarily large rotations has been presented, which will be summarized in this paper. Further time independent small strain plasticity will be included by making use of a layered model.

Geometry

To describe the shell geometry, use is made of a so-called basic triangle, with vertices that are material points on the shell middle surface (see Fig. 1). In the undeformed configuration this basic triangle is indicated as the Initial Reference Triangle (IRT), in a deformed configuration as the Current Reference Triangle (CRT).
The undeformed middle surface can be described by:

$$\boldsymbol{r} = \boldsymbol{r}_{\text{ref}} + \overset{*}{\boldsymbol{w}}, \tag{1}$$

in which $\boldsymbol{r}_{\text{ref}}$ specifies a point with Cartesian surface coordinates x_α on the IRT and $\overset{*}{\boldsymbol{w}}(x_\alpha)$ is a vector from the IRT to the middle surface. In this way an exact description of the undeformed geometry could be given. However, like in earlier papers [4, 9] an approximation is made by specifying only the component perpendicular to the IRT, denoted by $\overset{*}{w}$. Hence:

$$\overset{*}{\boldsymbol{w}} = \overset{*}{w}\boldsymbol{n}, \tag{2}$$

where $\boldsymbol{n}$ is the unit normal vector to the IRT.

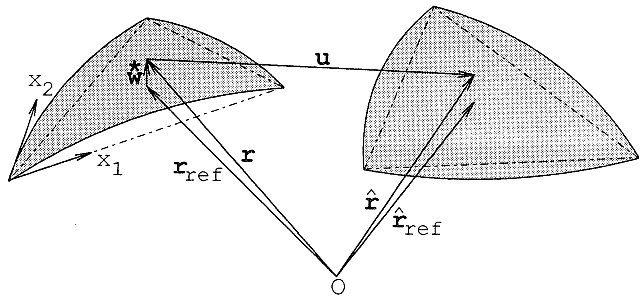

Fig. 1. Part of a shell in the undeformed and a deformed configuration.

Introducing a displacement vector $\boldsymbol{u}$, the deformed middle surface is given by:

$$\hat{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{u}. \tag{3}$$

For the description of the bending behaviour of the shell, use is made of a reference configuration, defined by:

$$\hat{\boldsymbol{q}} = \hat{\boldsymbol{r}}_{\text{ref}} + \overset{*}{w}\hat{\boldsymbol{n}}, \tag{4}$$

where $\hat{\boldsymbol{r}}_{\text{ref}}$ refers to the CRT and $\hat{\boldsymbol{n}}$ is the unit normal vector to the CRT. Provided that the strains remain small, this reference configuration can be seen as the undeformed configuration, moved as a rigid body to the current state.

Deformations

The deformations of the shell are characterized by membrane deformations and changes of curvature [10].
Using $\overset{*}{w}$ and the Cartesian components of the displacement vector $\boldsymbol{u}$, the membrane deformations can be expressed as [9] $(\alpha, \beta = 1 \ldots 2; i = 1 \ldots 3)$:

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + \overset{*}{w}_{,\alpha} u_{3,\beta} + \overset{*}{w}_{,\beta} u_{3,\alpha} + u_{\text{i},\alpha} u_{\text{i},\beta}), \tag{5}$$

where a comma denotes partial differentiation with respect to x_α.
Accepting errors of the order of magnitude inherent in thin shell theory, the changes of curvature can be written as [4]:

$$\varkappa_{\alpha\beta} = \tfrac{1}{2}(\hat{\varphi}_{\alpha,\beta} + \hat{\varphi}_{\beta,\alpha}), \tag{6}$$

with $\hat{\varphi}_\alpha$ as the rotations of the normal to the deformed middle surface with respect to the reference configuration. Using the Kirchhoff-Love hypothesis, $\hat{\varphi}_\alpha$ can be approximated by [4]:

$$\hat{\varphi}_\alpha = \hat{u}_{3,\alpha}, \tag{7}$$

where $\hat{u}_3$ is the displacement perpendicular to the CRT.

Constitutive equations

The derivation of the constitutive equations is based on the assumptions that the deformations remain small and that the state of stress is plane and parallel to the middle surface everywhere in the shell (see Fig. 2).
For small strain elasto-plasticity, the relation between the rates of the stresses $\sigma_{\alpha\beta}$ and the corresponding strains $\varepsilon_{\alpha\beta}$ can be written as [6, 12]:

$$\dot{\sigma}_{\alpha\beta} = (S_{\alpha\beta\xi\vartheta} - Y_{\alpha\beta\xi\vartheta})\dot{\varepsilon}_{\xi\vartheta}, \tag{8}$$

where $(S_{\alpha\beta\xi\vartheta} - Y_{\alpha\beta\xi\vartheta})$ represent the components of either the continuum tangent stiffness or the consistent tangent stiffness matrix. Aspects concerning the inclusion of the fraction model [1], the condition of a zero normal stress [7] and the way of integrating equation (8) will not be considered in this paper.
Using the Kirchhoff-Love hypothesis, the strains in a point (x_1, x_2, x_3) can be expressed in the membrane strains and changes of curvature in the point $(x_1, x_2, 0)$ on the middle surface:

$$\varepsilon_{\alpha\beta}(x_1, x_2, x_3) = \gamma_{\alpha\beta}(x_1, x_2, 0) - x_3 \varkappa_{\alpha\beta}(x_1, x_2, 0). \tag{9}$$

By substituting (9) into (8) and integrating over the thickness, one finds the relation between the rates of the stress resultants $N_{\alpha\beta}$ and $M_{\alpha\beta}$, and the deformations $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$:

$$\begin{vmatrix} \dot{N}_{\alpha\beta} \\ \dot{M}_{\alpha\beta} \end{vmatrix} = \int_{-h/2}^{+h/2} \begin{vmatrix} (S_{\alpha\beta\xi\vartheta} - Y_{\alpha\beta\xi\vartheta}) & -x_3(S_{\alpha\beta\xi\vartheta} - Y_{\alpha\beta\xi\vartheta}) \\ -x_3(S_{\alpha\beta\xi\vartheta} - Y_{\alpha\beta\xi\vartheta}) & x_3^2(S_{\alpha\beta\xi\vartheta} - Y_{\alpha\beta\xi\vartheta}) \end{vmatrix} \mathrm{d}x_3 \begin{vmatrix} \dot{\gamma}_{\xi\vartheta} \\ \dot{\varkappa}_{\xi\vartheta} \end{vmatrix} \tag{10}$$

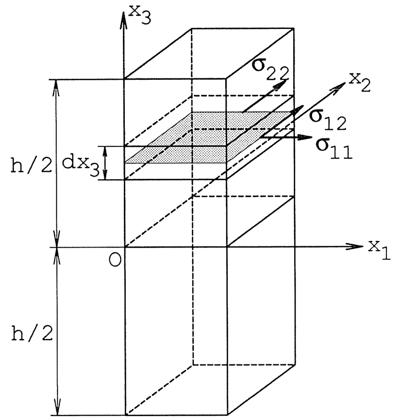

Fig. 2. Stress components at a distance x_3 from the shell middle surface.

Finite element formulation

The finite element formulation will be based on the principle of virtual work, which can be written as:

$$\delta W_{\mathrm{i}} = \delta W_{\mathrm{e}}, \tag{11}$$

where δW_{i} is the internal virtual work and δW_{e} is the virtual work of the external loads. For any part of the shell as discussed in the foregoing, δW_{i} is given by:

$$\delta W_{\mathrm{i}} = \int_{\hat{\mathrm{A}}} (N_{\alpha\beta}\delta\gamma_{\alpha\beta} + M_{\alpha\beta}\delta\varkappa_{\alpha\beta})\, \mathrm{d}\hat{A}, \tag{12}$$

in which the deformed surface is denoted by $\hat{A}$.
Further, δW_{e} is given by:

$$\delta W_{\mathrm{e}} = \int_{\hat{\mathrm{A}}} p_{\mathrm{i}}\delta u_{\mathrm{i}}\, \mathrm{d}A + \int_{\delta\hat{\mathrm{A}}} (V_{\mathrm{i}}\delta u_{\mathrm{i}} + m_{\alpha}\delta\varphi_{\alpha})\, \mathrm{d}\hat{s}, \tag{13}$$

in which p_{i}, δu_{i}, V_{i}, m_{α} and $\delta\varphi_{\alpha}$ are distributed surface loadings, virtual changes of the displacement components, forces per unit length, moments per unit length and virtual changes of the rotations of the normal to the deformed surface respectively. The boundary of the deformed surface is denoted by $\partial\hat{A}$.
It must be noted that $\delta\varphi_{\alpha}$ can be given by [4]:

$$\delta\varphi_{\alpha} = \delta\hat{\varphi}_{\alpha} + \delta\omega_{\alpha}, \tag{14}$$

where $\delta\omega_{\alpha}$ denotes virtual changes of the rotations of the normal to the deformed surface due to virtual rotations of the reference configuration.
The expressions (5), (6) and (7) can be added to the principle of virtual work by means of multiplier functions. Application of the standard variational principle offers the possibility of eliminating these multiplier functions. If the rotation component $\hat{\varphi}_{\mathrm{s}}$ along the boundary is further assumed to be given by (7), the virtual work equation can be written as [4]:

$$\begin{aligned}
&\int_{\mathrm{A}} \{\delta(N_{\alpha\beta}\tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + \overset{*}{w}_{,\alpha}u_{3,\beta} + \overset{*}{w}_{,\beta}u_{3,\alpha} + u_{\mathrm{i},\alpha}u_{\mathrm{i},\beta})) + \\
&\quad - \delta N_{\alpha\beta}\gamma_{\alpha\beta} + \delta(M_{\alpha\beta,\alpha\beta}\hat{u}_3) - \delta M_{\alpha\beta}\varkappa_{\alpha\beta}\}\, \mathrm{d}A + \\
&\int_{\partial\mathrm{A}} \{\delta(M_{\mathrm{nn}}\hat{\varphi}_{\mathrm{n}} + M_{\mathrm{ns}}\hat{u}_{3,\mathrm{s}} - (M_{\mathrm{nn,n}} + M_{\mathrm{ns,s}})\hat{u}_3)\}\, \mathrm{d}s = \\
&= \int_{\mathrm{A}} p_{\mathrm{i}}\delta u_{\mathrm{i}}\, \mathrm{d}A + \int_{\partial\mathrm{A}} \{N_{\mathrm{i}}\delta u_{\mathrm{i}} + M_{\mathrm{n}}\delta\varphi_{\mathrm{n}} + M_{\mathrm{s}}(\delta\omega_{\mathrm{s}} + \delta\hat{u}_{3,\mathrm{s}})\}\, \mathrm{d}s,
\end{aligned} \tag{15}$$

where the subscripts "*n*" and "*s*" refer to normal and tangential directions of the boundary. Note that the integrals will be evaluated over the undeformed surface in the current state, which is allowed due to the assumption of small deformations.
Equation (15) will be used as a starting point for the derivation of the discrete finite element equations.
Consider the finite element of Fig. 3. The degrees of freedom are 18 displacement components and 6 rotation-like scalars, connected to the element sides. These scalars provide inter element continuity of the bending moments and can for the linear case be identified with the rotations of the normal [9]. The displacements and the initial geometry are interpolated quadratically between the nodal points, while the rotation-like

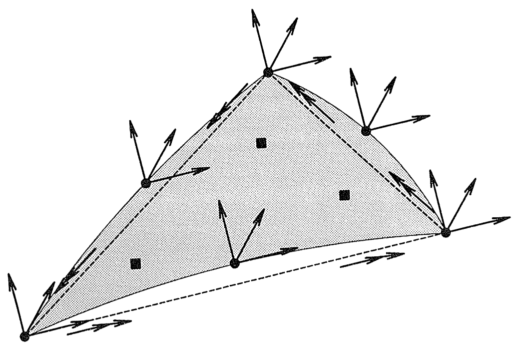

Fig. 3. Finite element with 6 nodal points, 3 material sampling points and 24 degrees of freedom.

scalars are interpolated linearly along each side between the corresponding vertices; the membrane stresses and the bending moments together with their dual deformations are interpolated linearly between the material sampling points.
Now the integrals in (15) can be evaluated explicitly, resulting in:

$$\delta(\boldsymbol{\sigma}_{\mathrm{m}}^{\mathrm{T}}\boldsymbol{E}_{\mathrm{m}}(\boldsymbol{u}_{\mathrm{e}})) - \delta\boldsymbol{\sigma}_{\mathrm{m}}^{\mathrm{T}}\boldsymbol{\varepsilon}_{\mathrm{m}} + \delta(\boldsymbol{\sigma}_{\mathrm{b}}^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{b}}^{\mathrm{u}}\hat{\boldsymbol{u}}_3 + \boldsymbol{D}_{\mathrm{b}}^{\varphi}\hat{\boldsymbol{\varphi}}_{\mathrm{n}}]) - \delta\boldsymbol{\sigma}_{\mathrm{b}}^{\mathrm{T}}\boldsymbol{\varepsilon}_{\mathrm{b}} = \boldsymbol{f}_{\mathrm{u}}\delta\boldsymbol{u}_{\mathrm{e}} + \boldsymbol{f}_{\varphi}\delta\boldsymbol{\varphi}_{\mathrm{n}}, \quad (16)$$

where $\boldsymbol{\sigma}_{\mathrm{m}}$, $\boldsymbol{\sigma}_{\mathrm{b}}$, $\boldsymbol{\varepsilon}_{\mathrm{m}}$, and $\boldsymbol{\varepsilon}_{\mathrm{b}}$ contain the membrane stresses, bending moments and their dual deformations respectively. $\boldsymbol{E}_{\mathrm{m}}$ is a function of the nodal displacements, which are collected in $\boldsymbol{u}_{\mathrm{e}}$. The nodal displacements $\hat{u}_3$ and the rotations $\hat{\varphi}_{\mathrm{n}}$ and φ_{n} are collected in $\hat{\boldsymbol{u}}_3$, $\hat{\boldsymbol{\varphi}}_{\mathrm{n}}$ and $\boldsymbol{\varphi}_{\mathrm{n}}$ respectively. Thanks to an appropriate definition of $\hat{\boldsymbol{u}}_3$ [4], the matrices $\boldsymbol{D}_{\mathrm{b}}^{\mathrm{u}}$ and $\boldsymbol{D}_{\mathrm{b}}^{\varphi}$ equal the ones found for total moderate rotations in [9].
Variations of $\boldsymbol{\sigma}_{\mathrm{m}}$ and $\boldsymbol{\sigma}_{\mathrm{b}}$ yield:

$$\boldsymbol{\varepsilon}_{\mathrm{m}} = \boldsymbol{E}_{\mathrm{m}}(\boldsymbol{u}_{\mathrm{e}}), \quad (17)$$

$$\boldsymbol{\varepsilon}_{\mathrm{b}} = \boldsymbol{D}_{\mathrm{b}}^{\mathrm{u}}\hat{\boldsymbol{u}}_3 + \boldsymbol{D}_{\mathrm{b}}^{\varphi}\hat{\boldsymbol{\varphi}}_{\mathrm{n}}. \quad (18)$$

Some special attention must be paid to the influence of the initial geometry on the membrane deformations. In [4, 9] it has been argued that in order to describe inextensional bending exactly, the term $\frac{1}{2}(\overset{*}{w}_{,\alpha}u_{3,\beta} + \overset{*}{w}_{,\beta}u_{3,\alpha})$ has to be linearized between the values in the vertices of the element. However, due to the fact that the bending behaviour of the element is approximated by that of a flat triangle, problems may arise in describing rigid body motions of an arbitrary patch of curved elements, due to the element-wise application of approximation (2). These problems can be circumvented by taking into account the influence of the initial geometry based only on average strains of the three curved element edges. At the moment this is one of the topics to be investigated in more detail.
For arbitrarily large rotations, it is impossible to express $\hat{\boldsymbol{\varphi}}_{\mathrm{n}}$ in $\boldsymbol{u}_{\mathrm{e}}$ and $\boldsymbol{\varphi}_{\mathrm{n}}$. Therefore, the expression for the time derivative of $\boldsymbol{\varepsilon}_{\mathrm{b}}$ has to be considered [3]. In [4] it has been shown that this rate equation is given by:

$$\dot{\boldsymbol{\varepsilon}}_{\mathrm{b}} = \boldsymbol{D}_{\mathrm{b}}^{\mathrm{u}}\boldsymbol{T}\dot{\boldsymbol{u}}_{\mathrm{e}} + \boldsymbol{D}_{\mathrm{b}}^{\varphi}\dot{\boldsymbol{\varphi}}_{\mathrm{n}}, \quad (19)$$

where it must be emphasized that the transformation matrix $\boldsymbol{T}$ depends solely on the displacements of the vertices of the element.
After collecting the deformations in $\boldsymbol{\varepsilon}$, the membrane stresses and bending moments in $\boldsymbol{\sigma}$, the degrees of freedom in $\boldsymbol{u}_\mathrm{f}$, the external loads in $\boldsymbol{f}$ and by making use of the constitutive equations (10), the discrete equations become:

$$\dot{\boldsymbol{f}} = (\boldsymbol{K} + \boldsymbol{G})\dot{\boldsymbol{u}}_\mathrm{f}, \tag{20}$$

$$\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}\dot{\boldsymbol{u}}_\mathrm{f}, \tag{21}$$

$$\dot{\boldsymbol{\sigma}} = (\boldsymbol{S} - \boldsymbol{Y})\dot{\boldsymbol{\varepsilon}}, \tag{22}$$

$$\boldsymbol{f} = \boldsymbol{D}^\mathrm{T}\boldsymbol{\sigma}. \tag{23}$$

Examples

The first example concerns a cylindrical shell roof subjected to a uniformly distributed self-weight load (see Fig. 4). The curved edges are simply supported, the straight edges are free. A small displacement elasto-plastic analysis has been performed, using an elastic-perfectly plastic Von Mises material. In thickness direction 5 layers have been applied, while further use has been made of the consistent tangent operator formulation. The calculation has been carried out under arc-length control, while the energy norm with $\varepsilon = 10^{-6}$ has been used to check on convergence.

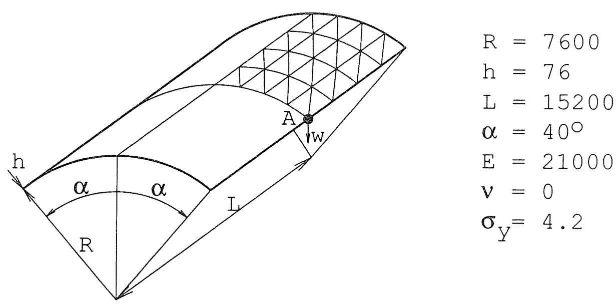

Fig. 4. Cylindrical shell roof; problem description and used finite element mesh.

Fig. 5 shows the vertical displacement at the center A of a free edge, depending on the load intensity. This result seems to be in good agreement with [6] and [11]. Further this figure contains the number of iterations required to obtain convergence for several load steps.
The second example concerns a clamped circular membrane with a uniform transverse load (see Fig. 6). In this analysis only geometrical nonlinearity has been taken into account. Because of the extremely low bending stiffness, the first loading step has to be chosen carefully. The calculations have been carried out using fixed step sizes with the same convergence checks as in the previous example.

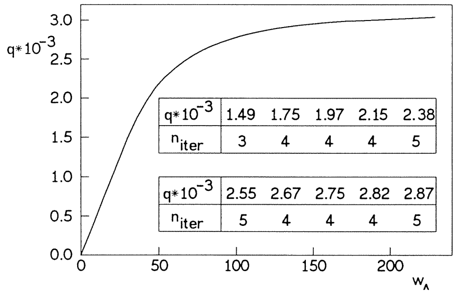

Fig. 5. Vertical displacement of point A versus the load intensity together with some convergence characteristics.

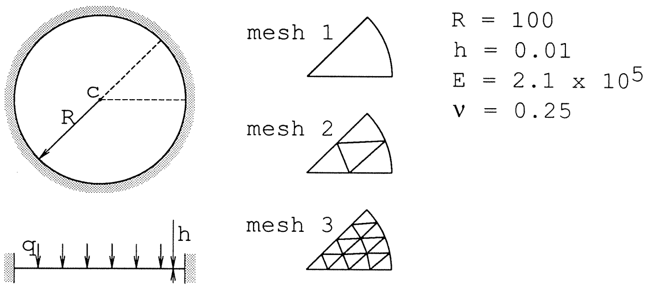

Fig. 6. Circular membrane; problem description and used finite element meshes.

Figs. 7a and 7b respectively compare the finite element solutions for the transverse displacement and the radial stress at the center of the membrane with analytical results of [13].
Examples concerning larger displacements and rotations can be found in [4] and [5].

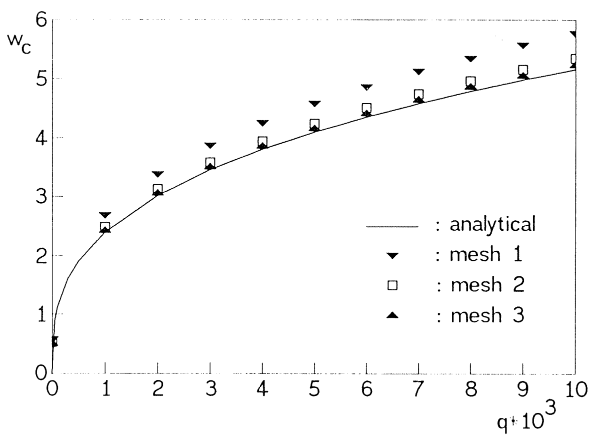

Fig. 7a. Finite element and analytical solutions for the vertical displacement of point C depending on the load intensity.

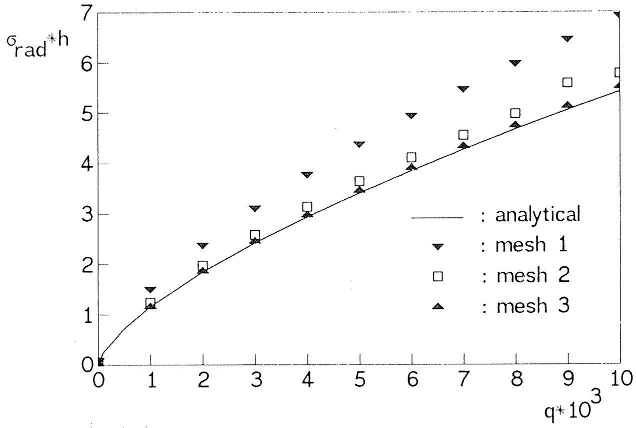

Fig. 7b. Finite element and analytical solutions for the radial stress at point C depending on the load intensity.

Conclusions

Starting from a flat plate with an initial deflection, a rather simple finite element formulation is given. Arbitrarily large rotations are taken into account by using a reference configuration. The introduction of material nonlinearities based on a layered model offers the possibility of making use of existing program code for plane stress analysis. Even for very thin shells accurate results can be found.

Acknowledgement

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