# Some aspects of artificial diffusion in flow analysis

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**1 Abstract**

A recent artificial diffusion method is briefly discussed. The method is illustrated in one and two dimensional convection diffusion problems. It is also used in an incompressible flow problem (Poiseuille flow). The incompressibility condition is imposed both weakly (weighted formulation) and with a discrete penalty formulation; results are compared for the Poiseuille flow. Some preliminary considerations on fluid-structure interaction are given. Finally, the question of whether to use direct or iterative solvers in flow problems is addressed.

**2 Preface**

For user friendly and efficient fluid-structure interaction modelling, it is desirable to have both the fluid part and the structure part available in one numerical package. Finite Element Methods are dominating structure modelling and are catching up in fluid modelling. This paper comprises a short literature survey and some of our first steps in implementing interaction problems in the DIANA finite element code.

**3 Galerkin/least-squares finite element methods**

Standard Galerkin finite element flow analysis encounters some problems:

- At high ratio of $u/\varepsilon$ ($u$ velocity, $\varepsilon$ viscosity) the boundary layer at the outflow boundary may be very small, corresponding with a large solution gradient. At such a high gradient the discretized solution field cannot approximate the exact solution field, and oscillations (wiggles) may be induced. These wiggles are not present in the exact solution of the continuum equations. They disappear at a small enough element size, and this size may be extremely small in practical calculations. Consequently, a numerical method is needed which gives reasonable solutions at finite element size.
- The incompressibility condition may lead to overly constrained elements. The solution field "locks", i.e. a nonphysical constant field is found. As opposed to the wiggly solution, locking does not disappear with decreasing element size. The same problem is encountered in the analysis of incompressible solids (rubberlike materials). To prevent locking the Babuska-Brezzi condition should be obeyed. In other words: the space of trial functions for velocity and pressures cannot be chosen independently. Examples of allowable spaces can be found in textbooks on finite element methods, Hughes [1].

Galerkin/least-squares finite elements are useful in tackling both problems, because,

- The wiggles are suppressed, Johnson [2].
- The velocity and pressure space can be chosen freely, and in particular they can be equal, Hughes [3].

Consider a differential operator $\boldsymbol{D}$ acting on a total column $\boldsymbol{u}$ with unknowns

$$\boldsymbol{D}\boldsymbol{u}=\boldsymbol{f} \tag{1}$$

where $\boldsymbol{f}$ is for instance an external force. Galerkin's method would read

$$(\boldsymbol{w}^{\mathrm{h}}, \boldsymbol{u}^{\mathrm{h}}) = \boldsymbol{L}(\boldsymbol{w}^{\mathrm{h}}) \tag{2}$$

in which $(\boldsymbol{v}, \boldsymbol{w})$ is a bi-linear inner product and $\boldsymbol{L}(\boldsymbol{v})$ is a linear form. Both the weighting functions $\boldsymbol{w}^{\mathrm{h}}$ and the trial functions $\boldsymbol{u}^{\mathrm{h}}$ belong to a restricted space $V^{\mathrm{h}}$. The suffix $h$ indicates that the space $V^{\mathrm{h}}$ is dependent on the mesh size $h$. The Galerkin/least-squares method adds an extra least-squares part

$$(\boldsymbol{w}^{\mathrm{h}}, \boldsymbol{u}^{\mathrm{h}}) + \int_{\Omega} \boldsymbol{D}\boldsymbol{w}^{\mathrm{h}}\boldsymbol{\tau}(\boldsymbol{D}\boldsymbol{u}^{\mathrm{h}} - \boldsymbol{f})\, d\Omega = \boldsymbol{L}(\boldsymbol{w}^{\mathrm{h}}) \tag{3}$$

As the space of trial functions is unaltered, the scheme is still consistent. The extra least-square term supplies stability in such a way that wiggles are suppressed. Consistency and stability together enable convergence of $\boldsymbol{u}^{\mathrm{h}}$ to the exact solution (thus indicating that locking is prevented). The matrix $\boldsymbol{\tau}$ is chosen so that it gives optimal convergence properties. In fact, it can be chosen in such a way that optimal convergence properties of each of the eigenvalues of the system are obtained, Shakib [4]. For an operator $\boldsymbol{D}$ with temporal derivatives, this means that the time coordinate must be treated as equivalent to the spatial coordinates, i.e. space-time finite elements should to be used. However, choosing the solution vector $\boldsymbol{u}^{\mathrm{h}}$ to be continuous for the time coordinate would imply that the solution early in time would also depend on the solution afterwards. Choosing the solution to be discontinuous in time circumvents this problem. The information is now propagated by imposing the solution vector of a space-time element weakly on the next element. Although quite different from the semi-discrete approach, this space-time scheme degenerates into the familiar Euler backward scheme for constant-in-time trial functions for the time coordinate. A finite element approach in both space and time allows for complete usage of finite element error estimates. This paper is concerned with time independent examples only.

*Remark 1*

The least-squares contribution contains second order derivatives (viscosity term). For linear elements, these second order derivatives vanish. For higher order elements, the second order terms may have to be incorporated.

*Remark 2*

A part of the least-squares term can be interpreted as adding artificial diffusion to the standard Galerkin scheme.

*Remark 3*
Note that the scheme also results from a Petrov-Galerkin method with weighting functions

$$w^{\mathrm{h}} + \boldsymbol{D}w^{\mathrm{h}}\boldsymbol{\tau} \tag{4}$$

## 4 One dimensional convection diffusion

Consider the boundary value problem

$$-\varepsilon u_{,\mathrm{xx}} + \beta u_{,\mathrm{x}} = 0; \qquad 0 \leq x \leq 1 \tag{5}$$

with boundary conditions

$$u(0) = 1, \qquad u(1) = 0 \tag{6}$$

The unknown $u$ represents, for instance, a temperature transported by diffusion and convective velocity $\beta$. Standard Galerkin reads

$$\varepsilon(u^{\mathrm{h}}_{,\mathrm{x}}, w^{\mathrm{h}}_{,\mathrm{x}}) + \beta(u^{\mathrm{h}}_{,\mathrm{x}}, w^{\mathrm{h}}) = 0 \tag{7}$$

for all $w^{\mathrm{h}} \in V^{\mathrm{h}}$. Here $(v, w) = \int_0^1 v(x)w(x)\mathrm{d}x$. Take $\varepsilon = 0.01, \beta = 1$, and $V^{\mathrm{h}}$ to be the space of 10 piecewise linear polynomials.
Because of the large element Peclet number $P = \beta h/\varepsilon = 10$, standard Galerkin yields a wiggly solution (Fig. 1).
Adding the least-squares term leads to

$$(\varepsilon + \tau\beta^2)(u^{\mathrm{h}}_{,\mathrm{x}}, w^{\mathrm{h}}_{,\mathrm{x}}) + \beta(u^{\mathrm{h}}_{,\mathrm{x}}, w^{\mathrm{h}}) = 0 \tag{8}$$

which clearly explains the expression "artificial diffusion". The parameter $\tau$ is chosen to be

$$\tau = \frac{h}{2\beta}\sqrt{\left(\frac{\alpha^2}{9 + \alpha^2}\right)}; \qquad \alpha = \frac{P}{2} \tag{9}$$

which is nearly optimal (in the sense of giving nodally exact results), Shakib [3]. See Fig. 2 for results with artificial diffusion.

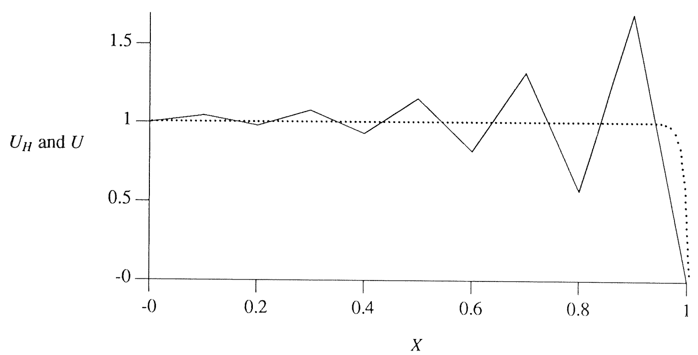


Fig. 1. Here $\beta = 1$ and $\varepsilon = 0.01$. Standard Galerkin. The exact solution is dotted.

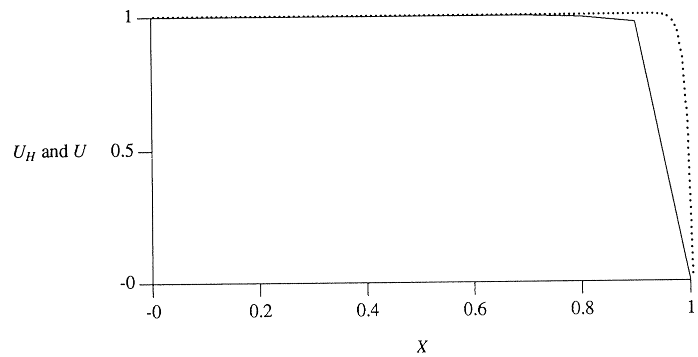


Fig. 2. Here $\beta = 1$ and $\varepsilon = 0.01$. With least-squares term. The exact solution is dotted.

## 5 Two dimensional convection diffusion

Consider the boundary value problem

$$-\varepsilon\,\Delta u + \boldsymbol{\beta}\nabla u = 0 \tag{10}$$

in which

$$\boldsymbol{\beta} = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \tag{11}$$

and in which $\Delta$ is the two-dimensional Laplace operator. For linear elements the Galerkin/least-squares approach gives,

$$\varepsilon(\nabla u^{\mathrm{h}}, \nabla w^{\mathrm{h}}) + \tau\boldsymbol{\beta}\boldsymbol{\beta}(\nabla u^{\mathrm{h}}, \nabla w^{\mathrm{h}}) + (\boldsymbol{\beta}\nabla u^{\mathrm{h}}, w^{\mathrm{h}}) = 0 \tag{12}$$

The eigendirections of the matrix $\tau\boldsymbol{\beta}\boldsymbol{\beta}$ are the streamline direction $\boldsymbol{\beta}$ and the perpendicular direction. In the streamline direction the eigenvalue is $\tau(\beta_1^2 + \beta_2^2)$ whereas the other eigenvalue equals 0. From this it becomes clear that the least-squares term effectively "adds diffusion" only in the streamline direction.

## 6 Incompressible Navier-Stokes

The incompressible (isotherm, time independent, no body forces) Navier-Stokes equations read:

$$\varrho\dot{\boldsymbol{u}}\nabla\dot{\boldsymbol{u}} - \varepsilon\,\Delta\dot{\boldsymbol{u}} + \nabla p = 0 \tag{13}$$

subjected to the incompressibility constraint

$$div\dot{\boldsymbol{u}} = 0 \tag{14}$$

with $\dot{\boldsymbol{u}}$ being the velocity, $p$ the pressure, $\varrho$ the density and $div\dot{\boldsymbol{u}} = \dot{u}_{1,1} + \dot{u}_{2,2} + \dot{u}_{3,3}$. The Galerkin/least-squares method is again useful in suppressing wiggles at high velocities.

Furthermore, it prevents elements with equally interpolated velocities and pressures from locking. Following Hansbo and Szepessy [5] we use the next method

$$\text{Galerkin terms} + \tau(\varrho\dot{\boldsymbol{u}}^{\mathrm{h}}\nabla\dot{\boldsymbol{u}}^{\mathrm{h}} + \nabla p^{\mathrm{h}}, \varrho\dot{\boldsymbol{u}}^{\mathrm{h}}\nabla\boldsymbol{w}^{\mathrm{h}} + \nabla q^{\mathrm{h}}) = 0 \tag{15}$$

where $\boldsymbol{w}^{\mathrm{h}}$ and $q^{\mathrm{h}}$ are the weighting functions for the velocities $\dot{\boldsymbol{u}}^{\mathrm{h}}$ and pressure $p^{\mathrm{h}}$ respectively. A Poiseuille flow is analyzed (length 5 m, height 1 m, velocity 1 m/s at nodes on the left edge, total fluid flow 0.75 $m^3$/s, viscosity 0.1 Ns/m, density 1 kg/$m^3$). Rectangular, four-noded elements are used (equal isoparametric interpolation of velocity and pressure; mesh size $h = 0.25$ m). A plot of the flow is given for $\tau$ once being 0 and once being chosen according to eq. 9 with $\beta$ the length of $\dot{\boldsymbol{u}}$.
It can be seen in Fig. 3 that without the extra least-squares term ($\tau = 0$) locking occurs (all velocities are found to be 1). This is prevented by activating the least-squares term (Fig. 4).

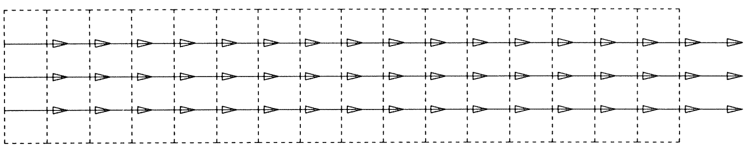

Fig. 3. Poiseuille flow without least-squares stabilization.

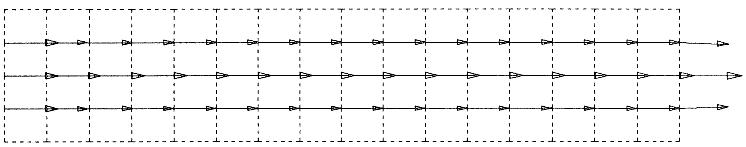

Fig. 4. Poiseuille flow with least-squares stabilization.

The velocity in the middle (2.5 m, 0.5 m) of the flow is found to be 1.128 m/s, whereas it should be 1.2 m/s. Also the total fluid flow at this section is found to be low: 0.71 $m^3$/s instead of 0.75 $m^3$/s. The incompressibility condition is not exactly satisfied. This is due to the fact that eq. 14 is only weakly imposed. Although this is not unforgivable (equilibrium will not be exactly satisfied either as it is also imposed weakly), we implemented a discrete penalty method. This scheme exactly yields 1.2 m/s for the velocity, as is to be expected since the incompressibility condition is imposed more strongly. Artificial diffusion of the type

$$\tau(\dot{\boldsymbol{u}}^{\mathrm{h}}\nabla\dot{\boldsymbol{u}}^{\mathrm{h}}, \dot{\boldsymbol{u}}^{\mathrm{h}}\nabla\boldsymbol{w}^{\mathrm{h}}) \tag{16}$$

is added to the discrete penalty formulation.

## 7 Fluid-structure interaction

Due to viscosity, a fluid particle at a solid wall will remain at the wall (no slip condition). The displacement of fluid particle and the wall will be the same. The nodal degrees of freedom of solid elements are displacements. Using displacement degrees of freedom for the fluid elements, enables automatic satisfaction of the no slip condition. It is also important to realize that the equivalent nodal forces for the solid and fluid elements should represent the same quantity for automatic coupling of fluid and solid elements. In the standard Galerkin formulation the nodal forces for both types of elements do agree on

$$\int_{\Omega} \sigma_{ij} \delta \dot{\varepsilon}_{ij} \, d\Omega \tag{17}$$

in which $\sigma_{ij}$ is the Cauchy stress tensor and $\varepsilon_{ij}$ the linear strain tensor. However, the extra artificial diffusion part in the fluid elements does disturb the equivalence of nodal forces. Fortunately, the fluid velocity near a wall is so small that artificial diffusion near a wall only plays a subordinate role in the equivalent nodal forces.
In this way fluid structure interaction is automatically accomplished (in geometrically linear analysis). In geometrically nonlinear analysis, an Arbitrary Lagrangian-Eulerian strategy is to be used.

## 8 Direct or iterative solvers

If the incompressibility condition is imposed weakly by introducing pressure degrees of freedom, some of the stiffness matrix diagonal terms are zero and pivoting is necessary to solve the equations. However, the least-squares operator stores small contributions on the diagonal terms, which means that pivoting was not required in the examples we studied. In the discrete penalty formulation, the zero diagonal terms do not show up in the first place. Iterative solvers may be used in large three-dimensional calculations to limit storage and cpu requirements. As the discrete penalty formulation causes an extreme increase in the matrix condition number, which is fatal for iterative solvers, the incompressibility condition should be imposed weakly. The small diagonal terms may have a devastating effect on the convergence properties of such solvers. A possible solution for this has been proposed by M. van Gijzen [6].

## 9 Summary

Artificial diffusion techniques are exemplified in convection-diffusion and incompressible flow problems. Discrete penalty elements are fatal for iterative solvers; however, if incompressibility is weakly imposed this will be at the expense of some leakage however. Choosing displacements for the nodal degrees of freedom in the fluid field facilitates easy fluid-structure interaction analysis.

## 10 Acknowledgement

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