Stress fluctuation mechanism of mesocrack growth, dilatancy and failure of heterogeneous materials in uniaxial compression

A.V. Dyskin
Department of Civil Engineering, The University of Western Australia, Nedlands, WA 6907, AUSTRALIA, Fax: (618) 9380 1044, E-mail: adyskin@cyllene.uwa.edu.au
On leave at Delft University of Technology, Faculty of Civil Engineering and Geo-Sciences

The fracture process in heterogeneous materials (e.g., concrete, rock, aggregate composites) under uniaxial compression proceeds in three stages differing in the scale levels involved. Stage 1 is the accumulation of wing cracks formed under compression from pre-existing defects. The wing cracks themselves cannot grow to an extent sufficient to cause failure. However, they produce an additional self-equilibrating and, in general, spatially random stress field. Therefore, some places can be found in the sample subjected to tensile stress components acting in the direction perpendicular to the direction of the applied compression and hence not getting suppressed by it. The magnitude of these tensile stresses increases as a square root of the wing crack concentration and, as the wing cracks accumulate, becomes sufficient to produce new tensile fractures. This starts Stage 2 in which these tensile fractures grow in a stable manner parallel to the direction of compression and become mesocracks (i.e., their sizes are now larger than the ones of the wing cracks, but still small compared to the sample dimensions). It is assumed that mesocracks grow in such a way to avoid wing cracks. Thus, the mean value of the part of the wing crack-produced random stress field that affects each mesocrack is positive (tensile), which eventually initiates the unstable grows of mesocracks and causes failure. This constitutes Stage 3 of the fracture process.

Key words: **spatially random stress field, Caussian Field, stress intensity factor, correlation radius**

1 Introduction

Three main modes of failure of heterogeneous materials under uniaxial compression are recognised (e.g., Paul, 1968): (a) *splitting* in which the sample gets broken into a number of columns parallel to the compression direction; (b) *shear (oblique) failure* in which one inclined macrocrack separates the sample into two and; (c) *spalling* in which small chips are ejected from the surface making the sample thinner and eventually producing its ultimate failure. The particular type of failure is supposed to be controlled the loading conditions (e.g., Paul, 1968; Peng and Johnson, 1972; van Vliet and van Mier, 1996) or by the material microstructure (e.g., Santiago and Hildsorf, 1973; Dyskin *et*

HERON, Vol. 43, No. 3 (1998) ISSN 0046-7316

al., 1996b, Sahouryeh and Dyskin, 1997). In all these cases it is apparent that the failure is ultimately caused by propagation of internal (pre-existing) cracks towards the direction of load (e.g., Brady, 1976; Atkinson, 1981; Allegre et al., 1982), some of which propagate so extensively that they eventually break the material. It is confirmed by numerous observations of cracks in sections made from samples loaded at different stress levels; the observations clearly demonstrate the increase in the crack length with the increase of loading (e.g., Peng and Johnson, 1972; Sangha et al., 1974).
Accordingly, a lot of efforts starting from the pioneering work by Griffith (1924) was directed towards modelling the failure mechanisms based on crack formation and propagation (e.g., Brace and Bombolakis, 1963; Hoek and Bieniawski, 1965; Brace et al., 1966; Fairhurst and Cook, 1966; Gol'dstein et al., 1974; Adams and Sines, 1978; Dey and Chi-Yuen Wang, 1981; Wittmann, 1981; Zaitsev, 1983; Nemat-Nasser and Horii, 1982; Galybin, 1985; Horii and Nemat-Nasser, 1985, 1986; Ashby and Hallam, 1986; Sammis and Ashby, 1986; Dyskin and Salganik, 1987; Germanovich and Dyskin, 1988; Fanella and Krajcinovic, 1988; Gramberg, 1989; Talonov and Tulinov, 1989; Ashby and Sammis, 1990; Cannon et al., 1990; Schulson et al., 1991; Dyskin et al., 1991, 1992, 1993, 1994a, b, 1996a, b; Germanovich et al., 1990, 1993, 1994a, b; Dyskin and Germanovich, 1993; Sadowski, 1994; Labuz et al, 1996; see also the review by Wang and Shrive, 1995).
Other concepts, not based on mechanisms of crack propagation, were also proposed assuming that the macrocrack formation is only a final stage of the failure process. First to be mentioned is the concept of the accumulation of possibly non-growing cracks or, in general, defects (e.g., Scholz, 1968) with the following localisation (due for example to their interaction, e.g., Horii, 1993; Okui and Horii, 1994; Bazant and Xiang, 1997) and coalescence into a macroscopic fracture (e.g., Paul, 1968; Dey and Wang, 1981; Wong, 1982; Du and Aydin, 1991; Lockner et al., 1992; Ashby and Hallam, 1986; Krajcinovic, 1996; Blechman, 1997a, b). There is however a problem in visualising the mechanism of crack coalescence, since even in tension cracks try to avoid each other (Melin, 1983). It is even more difficult in 3-D where cracks do not necessarily coplanar (see also discussion in Germanovich et al., 1994a). Bazant and Xiang (1997) considered a mechanism based on forming shear bands consisting of slim columns (beams) by locally growing cracks. This beams subsequently buckle causing the failure. This model is also 2-dimensional; it is not clear why in 3-D the growing cracks are going to arrange themselves not only parallel but also close to each other to form slim columns.
Another mechanism being considered is strain localisation due to local post-peak softening of the material (e.g., Papanastasiou and Vardoulakis, 1994; Vardoulakis and Sulem, 1995). This approach merely transforms the difficulties of finding the mechanism of deformation and fracture to a microscale.
The advance in computers made possible direct simulations of the interaction of numerous elements of material microstructure. In compression this was performed by considering the interaction between many wing cracks (e.g., Horii, 1993; Okui and Horii, 1994), by modelling the microstructure as lattice or beam networks (e.g., van Mier, 1992) or as assemblies of random particles with different types of contact interaction, which are variants of the distinct element method (e.g., Bazant et al., 1990). (Blair et al, 1993) conducted computer simulation of failure in compression which accounted for the random stress field originated from the material heterogeneity by introducing a stress perturbation term (it was not clear from the paper whether the equilibrium equations were kept satisfied after the random perturbance).

In the present paper mechanisms of crack propagation are considered and a 3-stage, three-dimensional model is proposed.

2 Mechanisms of crack initiation and growth in uniaxial compression

In perfectly homogeneous materials, uniaxial compression does not cause any tensile stresses, therefore no crack generation or growth are possible. Hence, the observed phenomena of crack growth significantly depend on the material heterogeneity. The elements of material heterogeneity (inhomogeneities, pores, cracks, etc.) act as stress concentrators producing local stress redistribution, generating local tension and thus initiating local failure and crack growth.

In compression, the role of initial, pre-existing cracks was investigated by (Brace and Bombolakis, 1963; Hoek and Bieniawski, 1965; Brace et al., 1966; Fairhurst and Cook, 1966; Dey and Chi-Yuen Wang, 1981; Nemat-Nasser and Horii, 1982; Horii and Nemat-Nasser, 1985, 1986; Ashby and Hallam, 1986; Dyskin and Salganik, 1987; Germanovich and Dyskin, 1988; Talonov and Tulinov, 1989; Ashby and Sammis, 1990; Cannon et al., 1990; Schulson et al., 1991; Theocaris and Sakellariou, 1991; Dyskin et al., 1991, 1994a, b; Germanovich et al., 1990, 1993, 1994a; Dyskin and Germanovich, 1993; Li and Nordlung, 1993; Baud et al, 1996), the role of pores by (Gol'dstein et al., 1974; Zaitsev, 1983; Galybin, 1985; Sammis and Ashby, 1986; Isida and Nemat-Nasser, 1987; Kemeny and Cook, 1991; Dyskin et al., 1992, 1993) and of stiff inhomogeneities (e.g., Bazant and Xiang, 1997; Blechman, 1997a, b). The initial, pre-existing cracks with contacted faces can be considered as the strongest source of the secondary crack growth, at least in comparison with pores (Dyskin et al., 1992, 1993).

The main feature of the crack growth in uniaxial compression, as emerged from the 2-D experiments and analyses, is their capacity for extensive growth (at least for initial cracks and pores) which has been believed to be the main mechanism of splitting.

The majority of cited works were devoted to studying the 2-D case mainly because of enormous technical difficulties related to 3-D studies. However, with the advance of experimental techniques the 3-D studies became possible. Adams and Sines (1978) used a transparent plastic (polymethyl-methacrylate) with disk-like flaws created inside the sample by cutting it into two blocks, machining semi-circular inclined slots into each block and then cementing them together. Only restricted crack growth was observed which could have been attributed to insufficient dimensions of the sample. However, further experiments on growth of internal 3-D cracks in sample from various materials (Dyskin *et al.*, 1994a, b; Germanovich *et al.*, 1994a, b) sufficiently large compared to the initial crack sizes showed that it is an intrinsic feature of the 3-D wing cracks that they are not capable of extensive growth in compression sufficient to cause failure.

Nevertheless, in large numbers, the wing cracks produce new tensile fractures or cracks extensively growing towards compression produced by the superposition of additional stresses generated by the wing cracks at places where the components acting in the directions perpendicular to the compression axis are tensile (in these directions they are not suppressed by the applied compression, Dyskin et al., 1996a). These stress components are self-equilibrating in the sense that their average is zero (Blair et al, 1993; Dyskin, 1997; Bazant and Xiang, 1997) and most probably random (Dyskin, 1997) due to the random nature of the wing crack locations and dimensions.

The theory of crack growth caused by spatially random stress fields was proposed by (Dyskin, 1997) for the case when the crack is created by the random stress fluctuation (or evolves from a pre-existing microcrack which dimensions are not high as compared to the correlation radius of the stress fluctuations). In this case the crack will obviously be created at the place of the maximum stress. This was shown to result in a non-zero total force associated with the stress fluctuations (even self-equilibrating), which constitutes a special mechanism of crack growth (it should be noted that for a large crack independently located in the random stress field, the stress fluctuations will be averaged out, so the total force associated with the stress fluctuations will be zero).

3 Mechanism of failure in uniaxial compression

On the basis of previous experimental and theoretical analysis it is possible to envisage the following mechanism of failure in uniaxial compression (see also Dyskin and Sahouryeh, 1997). Failure preparation starts with the formation and accumulation of wing cracks (Figure 1) producing an additional stress field which is obviously self-equilibrating and, hence, its normal components are compressive in some areas and tensile in others. This will be called *Stage 1* of fracture process in compression.

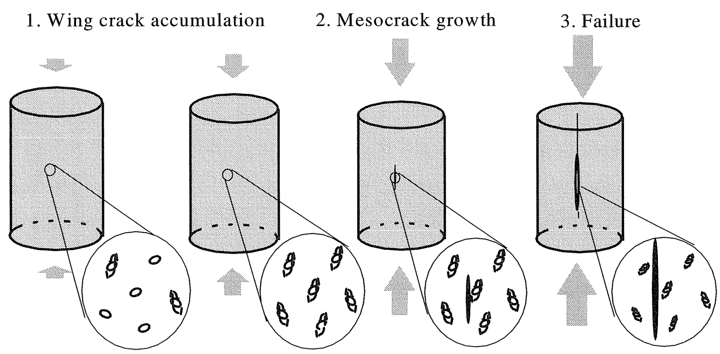

Fig. 1. Three-stage mechanism of fracture in uniaxial compression (after Dyskin and Sahouryeh, 1997)

The tensile stresses in the directions perpendicular to the compression axis do not get suppressed by the applied axial compression. Hence, as the loading proceeds and the number of wing cracks increases, these tensile stresses get stronger and can form large tensile fractures, macroscopic with respect to wing cracks, but still small compared to the sample dimensions. They will be called *mesocracks*. This starts *Stage 2* which also covers the phase of stable growth of mesocracks. Further loading will lead to unstable growth of mesocracks and failure. This unstable phase constitutes *Stage 3*, the formation of macroscopic failure. It seems that it is this stage at which the particular failure mode (splitting or shear failure) becomes apparent.

Thus the role of Stage 1 is to produce the wing cracks in sufficient concentrations to cause the initiation and growth of mesocracks. At Stage 2, as the mesocracks grow, they open and produce dilatancy (inelastic increase in the sample volume, e.g., Brace et al., 1996), the effect which previously was attributed to the wing cracks themselves (e.g., Dyskin and Salganik 1987; Germanovich and Dyskin 1988; Dyskin et al, 1991; Kemeny, 1991).

4 Model of dilatancy and fracture in uniaxial compression

4.1 *Stage 1. Stresses generated by wing cracks*

The wing cracks are formed from sliding pre-existing cracks. In the case when a pre-existing crack is disk-like, the developed wing crack may look like shown in Figure 2 which demonstrates the results of a uniaxial compressive loading of a transparent sample with an internal crack (Dyskin et al., 1994a). The picture corresponds to the maximum achievable size of the wing crack (i.e., the situation immediately before the load reached the strength of the matrix).

Since the wing cracks themselves cannot grow extensively they do not cause failure. Furthermore, because their dimensions are much smaller than the dimensions of the mesocracks their opening is also much smaller therefore the wing crack contribution to dilatancy can be neglected as compared to the one of the mesocracks. It should be noted that the wing crack formation produces acoustic emission, however at Stage 2 (the stage of mesocrack growth) the cumulative number of acoustic events is known to be proportional to dilatancy (Scholtz, 1968; Sano et al., 1981). The analysis shows (Dyskin, 1989) that this acoustic emission is related to the mesocrack growth. The proposed modelling therefore only concerns with the determination of statistical properties of the stress fluctuations generated by the wing cracks, which is required to model the mesocrack growth and opening.

The determination of statistical properties of the stress generated by defects or wing cracks is generally complicated by the necessity to account for the interaction between many defects. Calculating the interaction presumes that each defect should be considered as being loaded by a superposition of two stress fields: the external (original) stress field and an additional one which is the sum of stress disturbances introduced by all other defects in the continuum material on the position of the defect in question.

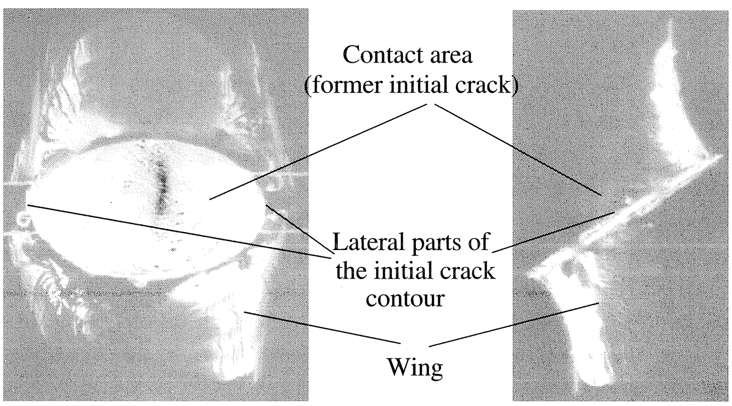

Fig. 2. The wing crack (after Dyskin et al., 1994a).

For wing cracks however, the influence of this additional stress field is minor. This is because the wing cracks are developed from pre-existing initial shear cracks inclined to the compression axis with both shearing over the initial cracks and the resulting wing opening mainly controlled by the high axial compression. Therefore, the change in the opening associated with the influence of additional stresses produced by other cracks can, in the first approximation, be neglected (due to friction between the surfaces of the initial cracks, it is unlikely that the stress fluctuations can produce any substantial changes in the shearing). As a result, the random stress fluctuations can be computed as a superposition of the additional stresses generated by randomly located defects each being considered as subjected only to the original stress field.

The modelling will be based on calculating the random stress field as the superposition of the fields produced by individual non-interacting wing cracks. In order to avoid technical difficulties the wing cracks will be modelled by point defects producing asymptotically, at large distances, the same stress field as the original wing cracks. This is equivalent to representing the wing crack as a combination of force dipoles (e.g., Dyskin and Mühlhaus, 1995), Figure 3.

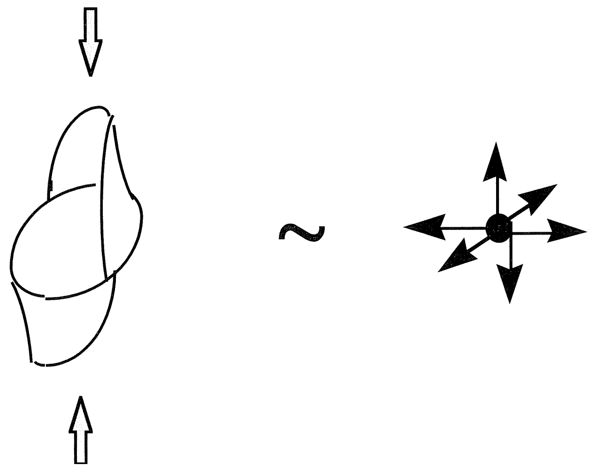

Fig. 3. Modelling of a wing crack with a combination of force dipoles.

This approximation, called the dipole asymptotics (e.g., Dyskin and Mühlhaus, 1995; Dyskin *et al.*, 1996a), has a strong non-integrable singularity at the origin, which requires the introduction of *excluded volumes* surrounding each wing crack whose size is an additional parameter. In general, this is an artificial construction, since real cracks do not produce non-integrable singularities. Furthermore, when the mesocrack growth is considered, the wing crack-generated stress fluctuations should be computed at all points including the ones close to wing cracks. In this case the errors associated with the singularity in the dipole asymptotic stress representation can be large. There is nevertheless a case when the simplification offered by the dipole asymptotics for calculating the influence of the stress fluctuations on the growth of a mesocrack is still possible. This is the case of mesocracks growing in such a way that they avoid the wing cracks thus effectively *forming* excluded volumes where the stress field does not have to be evaluated.

The existence of this type of excluded volumes still remains to be confirmed, but there is already some information indicating that such a situation is possible at least for the mechanism of mesocrack growth in uniaxial compression. Germanovich *et al.* (1994b) (see also Dyskin *et al.*, 1994b) have found from testing of transparent samples with internal disk-like cracks in uniaxial compression that if one 3-D crack is located in the path of another, the latter will neither get through, nor get arrested as routinely observed in 2-D tests (eg, Horii and Nemat-Nasser, 1985). Instead, it will merely flow around the crack-obstacle and proceed further. Cut-sections of rocks (eg, Peng and Johnson, 1972) also show traces of large cracks growing in the direction of uniaxial compression avoiding small ones.

These observations prompt a hypothesis that the wing cracks are surrounded by *excluded volumes* which are avoided by the mesocrack in the course of its growth. Then, if the wing crack-generated random stress field is considered from the point of view of its effect on the mesocrack propagation, only the points that are outside the excluded volumes should be taken care of. In the following simplified consideration the excluded volumes will be assumed spherical with the radius a equal to the characteristic size of the wing crack , i.e. to the radius of the initial crack.

The dipole asymptotics for wing crack in an isotropic material, is expressed in coordinate set (x_1,x_2,x_3) with the origin at the crack centre as follows (Gol'dstein and Kaptsov, 1982)

$$\Delta\sigma_{ij}(\xi) = -\frac{(1-2\nu)\mu}{4\pi(1-\nu)\xi^3}\left\{ \delta_{ij}\left[\frac{1-4v}{1-2v}u_{kk} - 3W\right] \right. \\ \left. +\xi_{,i}\xi_{,j}\left[\frac{15W}{1-2v} - 3u_{kk}\right] - \frac{6v}{1-2v}[\xi_{,j}\xi_{,k}u_{ki} + \xi_{,j}\xi_{,k}u_{kj}] - 2u_{ij} \right\} \tag{1}$$

Here $W = u_{ik}\xi_{,i}\xi_{,k}$, $\xi = |\xi|$, u_{ik} is the symmetrical tensor of dipole moments (eg, Dyskin *et al.*, 1996a), ",i" is derivative with respect to x_i and summation is presumed over repeated indices, μ is the shear modulus, ν is Poisson's ratio.

The components of the tensor of dipole moments will be found for the case when the wing cracks are fully developed, i.e. they have reached their maximum dimensions. A model proposed by Dyskin *et al.* (1996a) will be used according to which the maximum total volume of opening of both wings produced by pre-existing cracks is

$$V_{wing} \cong pa^3\frac{(2-\nu)}{(1+\nu)^2\mu} \tag{2}$$

where a is the radius of the initial crack, Figures 2, 3.

Then, taking into account that the wing cracks grow axisymmetrically with respect to the direction of compression, the corresponding component of the tensor of dipole moments is assumed equal to the maximum total volume of opening of both wings produced by pre-existing cracks (Dyskin *et al.*, 1996a), averaged over all inclinations of initial cracks to the loading directions. This gives

$$u_{ik} = \delta_{ik}(1-\delta_{k3})V_{wing}/\pi \tag{3}$$

Expressions (1)–(3) allow calculating the statistical properties of the stress field outside the excluded volume.

4.2 *Statistical properties of wing crack - generated stresses*

Consider the stress field generated by wing cracks located at homogeneously random points ξ_α in a volume V. The stress tensor at a point x is

$$\sigma(x) = \sigma^0 + \sum_\alpha \sigma_v(x, \xi_\alpha) \tag{4}$$

Here σ^0 is the stress which would be created by the external load in the absence of wing cracks, let this stress field be uniform, and $\sigma_V(x, \xi_\alpha)$ is the additional stress generated by a wing crack located at a point ξ_α in the volume V (here $s_V(x, \xi_\alpha)$ is the exact value; its dipole asymptotics will be used later).

Suppose the interaction between the wing cracks can be neglected. Then for the case of equal excluded volumes, V_0, the expected value is (due to the statistical homogeneity it is sufficient to consider stresses only at the origin)

$$\begin{aligned}\langle\sigma\rangle &= \sigma^0 + \frac{M}{V - V_0}\int_{V-V_0} \sigma_v(0, \xi)dV_\xi \\ &= \sigma^0 + \frac{M}{V - V_0}\left[\int_V \sigma_v(0, \xi)dV_\xi - \int_{V_0} \sigma_v(0, \xi)dV_\xi\right]\end{aligned} \tag{5}$$

where M is the number of wing cracks in volume V. As $V \to \infty$, since the integral over V can be set equal to zero (e.g., Germanovich and Dyskin, 1994c), (5) assumes the form

$$\langle\sigma\rangle = \sigma^0 - N\int_{V_0} \sigma(\xi)dV_\xi \tag{6}$$

where N is the number of wing cracks per unit volume.

For low wing crack concentrations when the influence of the excluded volumes on their distribution is negligible, the random variables ξ_α can be presumed statistically independent. Then the correlation function, $B(x) = <\sigma(0)\sigma(x)>-<\sigma>^2$, assumes the form

$$B(x) = \langle\sigma(0)\sigma(x)\rangle - \langle\sigma\rangle^2 = N\int_{V - V_0} \sigma(\xi)\sigma(\xi - x)dV_\xi \tag{7}$$

For the sake of simplicity the excluded volumes are assumed to be spheres of radius a. Let the uniaxial compression be applied in the x_3 direction. Only one component of the additional stresses, σ_{11} will be considered (the correlation between different component is neglected). Hereafter, σ, B mean σ_{11}, B_{11}.

After substituting (2) and (3) into (1) and then the results into (6), (7) one can find the mean, $<\sigma>$, and the variance, $B(0)$:

$$\langle\sigma\rangle = \frac{2(2 - \nu)\Phi(\nu)}{15\pi(1 - \nu)(1 + \nu)^2}pw, \quad B(0) = \frac{(2 - \nu)^2 I(\nu)}{4\pi^3(1 - \nu^2)^2}p^2 w \tag{8}$$

where $w = Na^3$ is the dimensionless concentration of the excluded volumes which, in accordance to the way they are chosen, is approximately equal to the concentration of initial (pre-existing) cracks, p is the magnitude of the applied compression,:

$$\Phi(v) = 3(3+5v), I(v) = 4(4v^2+24v/7+39/7)/15 \tag{9}$$

It should be emphasised that because the mesocracks grow avoiding the excluded volumes, $<\sigma>$ is positive, so the mesocracks are in average subjected to uniform tensile stress in the directions perpendicular to that of compression. This stress will be called the *background tensile stress*. Its magnitude, according to (8), is proportional to the concentration of initial (pre-existing) cracks. The magnitude of the stress fluctuations characterised by their standard deviation, $B(0)^{1/2}$, is proportional to square root of the concentration of initial cracks.
Another important statistical characteristics of the random stress field is the average correlation radius which is the size of an area where the statistical dependence is essential

$$\rho = \langle \rho(\eta) \rangle, \rho(\eta) = B(0)^{-1} \int_0^X B(t\eta)dt \tag{10}$$

Here averaging is performed over all orientations of the unit vector η.
For the considered case (see Dyskin, 1997)

$$\rho = J(v)a, J(v) \approx 0.863 - 0.216v \tag{11}$$

where a is the radius of the initial crack.
It is seen that the correlation radius is of the order of the wing crack size.

4.3 *Mesocrack initiation*

It will now be assumed that the random stress field generated by the wing cracks can be approximated by the Gaussian one, i.e. its statistical behaviour can completely be determined by the tensors $<\sigma>$ and $B(\mathbf{x})$.
Suppose the stress magnitude is sufficient to produce local fractures (cracks) somewhere in the material. Obviously this will first (i.e., at the minimal external load) happen at a location where the magnitude of stress, $\sigma(0)>0$, is a maximum. Let us put the origin of a Cartesian coordinate frame at this place. In a vicinity of the origin, the stress distribution will be determined by $\sigma(0)$. The average distribution of the corresponding stress component in the intact material on the plane (x_1, x_3) where the crack will be located has the form (e.g., Feller, 1971):l

$$\langle \sigma(\mathbf{x}) | \sigma(0) \rangle = \langle \sigma \rangle + \langle \Delta\sigma(\mathbf{x}) \rangle, \langle \Delta\sigma(\mathbf{x}) \rangle = [\sigma(0) - \langle \sigma \rangle] B(\mathbf{x})/B(0) \tag{12}$$

where $<\cdot|\cdot>$ stands for the conditional mathematical expectation and $\mathbf{x} = (x_1, x_3)$.
When the crack is initiated, the problem of determining its opening can formally be solved by considering the crack loaded by tractions equal, with inverse sign, to the corresponding stress components acting in the original material. This means that there is no direct back influence of the crack

on the original stress field, unless the crack affects the sources of the stresses (wing cracks). Only the case when the crack's influence on the stress-generating wing cracks (or, generally, defects) can be neglected will be considered here.

As the crack grows, it is subjected to additional tractions having a mean $\langle\Delta\sigma(x_2, x_3)\rangle$, in excess of the usual mean stress $\langle\sigma\rangle$. This additional load is solely due to the special location from which the crack has evolved and is a result of stress fluctuations.

It should be noted that in principle there might also be additional shear tractions acting on the crack if at least one shear component is correlated with the normal one. Then the mechanism and direction of crack growth will be affected. However, in the considered case the direction of crack growth is determined by the direction of the applied compression. Therefore the influence of the shear traction can in the first approximation be neglected.

In order to make the following analysis possible the correlation function will be presumed to vanish strongly enough as $|\mathbf{x}| \to \infty$, so, in average, the crack can be modelled as a crack loaded by uniform stress $\langle\sigma\rangle$ and a pair of concentrated forces (see Dyskin, 1997) with the magnitude

$$F = \int_{R^2} \langle\Delta\sigma(\mathbf{x})\rangle dx_1 dx_2 = \frac{\sigma(0) - \langle\sigma\rangle}{\sqrt{B(0)}}\kappa,\ \kappa = \frac{1}{\sqrt{B(0)}}\int_{R^2} B(\mathbf{x}) dx_1 dx_2 \tag{13}$$

For the considered case (see Dyskin, 1997)

$$\kappa = \Psi(\upsilon) a^2 B(0)^{1/2},\ \Psi(\upsilon) \approx 5.247 - 1.196\,\upsilon \tag{14}$$

4.4 *Stage 2. Mesocrack growth*

When the mesocrack is initiated, it first grows under the action of the tractions with the mathematical expectation (12) which can approximately be modelled by the action of a pair of concentrated forces (13) applied to the opposite mesocrack faces. Then in the course of its stable growth, the mesocrack will pass near several regions of high local tensile stresses. It is hypothesised here that the mesocrack will deviate or branch or sprout new cracks in order to pass through the nearest region with the maximum possible tensile stresses, Figure 4a. This will produce a tongue comparable in size with the correlation radius, ρ, of the stress fluctuations, which will then be smoothened by additional growth of other parts of the crack contour, Figure 4b. Thus, a larger approximately disk-shaped crack will be formed. This will result in adding a new pair of concentrated forces each time the crack passes through the chosen region. Since at each step the crack chooses only one region (the probability of two or more regions simultaneously happening on the crack path is neglected), the total number of concentrated forces will be proportional to the crack size.

The values of the concentrated forces can be computed from (13) if the stress fluctuations are known. By assumption, each step in the mesocrack propagation involves choosing the region with maximum fluctuation (the region area is $\sim\rho^2$) from the vicinity of the current crack contour (the total area of $2\pi r\rho$). This corresponds to choosing a maximum from $m \sim 2\pi r/\rho$ normally distributed random stress disturbances, $\Delta\sigma_1, \ldots, \Delta\sigma_m$, which can be assumed independent because they belong to the areas more than the correlation radius apart from each other.

Suppose the mesocrack radius is sufficiently large as compared to the average correlation radius, ρ. Then m is large. It is known (e.g., David, 1970) that the random variable

$$(2\ln m)^{1/2}(\Delta\sigma_{\max}B(0)^{-1/2} - l(m)), \text{ where } l(m) = \sqrt{2\ln m} - \frac{\ln(\ln m) + \ln 4\pi}{2\sqrt{2\ln m}} \tag{15}$$

has asymptotically, for large m, the distribution function $\Lambda_3(x) = \exp(-\exp(-x))$. Hence, the mathematical expectation of the maximum stress, $\Delta\sigma_{(m)}$, will have the form

$$\langle\Delta\sigma_{(m)}\rangle = l(m) = \sqrt{2\ln m} - \frac{\ln(\ln m) + \ln 4\pi}{2\sqrt{2\ln m}} \tag{16}$$

Then, using (16) and keeping only the leading asymptotic term, one has

$$E(\sigma(0)_{(\max)}) - \langle\sigma\rangle \sim \sqrt{2B(0)\ln 2\pi r/\rho} \tag{17}$$

This value should now be used in (14) instead of $\sigma(0) - \langle\sigma\rangle$.
For the sake of simplicity, the mesocrack will be modelled by a disk-like crack loaded by the uniform load $\langle\sigma\rangle$ and additional tractions modelling the concentrated forces, Figure 4c. Since only one pair of forces is added at each step, the average magnitude of the additional tractions acting at the new surface, $2\pi r \cdot (\rho/2)$, will be $F/\pi\rho r$. At each step from all $2\pi r/\rho$ regions at the crack perimeter only the one with the maximum stress is chosen. Hence the tractions are (κ is given by (14)):

$$q(r) = \frac{F(r)}{\pi\rho r}, F(r) = \kappa\sqrt{2\ln\frac{2\pi r}{\rho}} \tag{18}$$

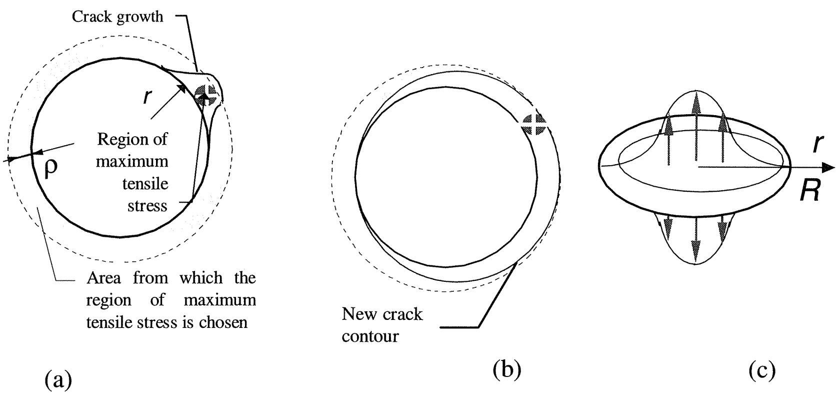

Fig. 4. Mechanism of mesocrack growth: (a) first the crack grows towards the region with maximum tensile stress; (b) other parts of the crack grow to smoothen the shape; (c) the model.

Because the logarithmic dependence (18) is weak, $F(r)$ can be approximated by its value for $r = R$. Then, by employing the solution for a disk-like crack loaded by concentrated forces of unit intensity distributed over a circumference of radius r (e.g., Tada, *et al.*, 1985) and integrating it from 0 to R with the weight $F(R)/\pi\rho r$, one obtains the *average* stress intensity factor.

The proposed usage of the average stress intensity factor needs some comments. Indeed, it is conventional to consider criteria of crack growth in terms of maximum values of the stress intensity factors. One should keep in mind though that this corresponds to 2-D cracks, in which case the end with the higher values of the stress intensity factors may be the first to satisfy the chosen criterion and hence ensure the crack growth.

For the *stable* growth of real 3-D cracks the situation is however different. Suppose, the criterion of crack growth gets satisfied at a certain point of the crack contour (point A at Figure 5a). Then a new surface will be produced in a vicinity of that point, Figure 5b. Because of the assumed stable character of the crack growth this new surface will remain to be a small portion of the total crack surface contributing little to the overall crack extension. In order to produce a noticeable crack extension, the criterion should get satisfied at a considerable part of the crack contour, which is not attainable if the criterion is formulated in terms of maximum stress intensity factors.

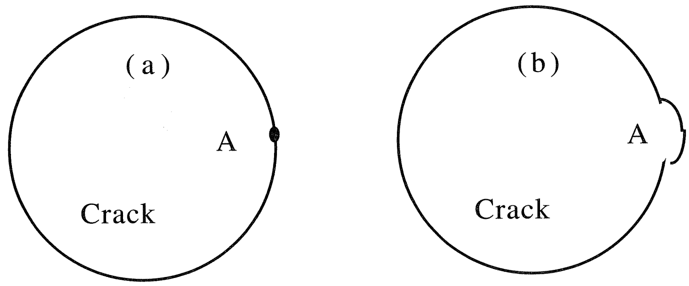

Fig. 5. Stable crack growth according to conventional criteria involving maximum values of stress intensity factors: (a) the growth criterion is first satisfied at a point (point A) of the contour; (b) the possible crack extension near that point contributes little to the crack extension.

A simple approximation which could take into account the requirement that the criterion of crack growth is satisfied at a considerable part of the crack contour would be to formulate the criterion in terms of the stress intensity factors averaged over the crack circumference. A further simplification would be to replace this spatial averaging with the averaging over realisations (mathematical expectation) of the random stress field (since the problem of stress concentration is linear, this simplification can be justified by assuming the ergodicity of the random stress field which is the equivalence between the spatial averaging and averaging over realisations).

As a result, the equation for determining the critical crack radius, R_{cr} (the radius corresponding to the onset of unstable crack growth found from the condition $d\langle K_I\rangle/dR = 0$) will have the form:

$$\langle K_1\rangle = \frac{F(R)}{\rho\sqrt{\pi R}} + 2\langle\sigma\rangle\sqrt{R/\pi},\ R_{cr} = \frac{F(R_{cr})}{2\rho\langle\sigma\rangle},\ \langle K_I\rangle = K_{Ic} \qquad (19)$$

The criterion of unstable crack growth assumes the form:

$$\frac{8\langle\sigma\rangle F(R_{cr})}{\pi\rho} = K_{Ic}^2 \tag{20}$$

where K_{Ic} is the fracture toughness of the matrix between the mesocracks.
This approximation also allows calculating the volume of crack opening, i.e. the crack contribution to dilatancy. The calculations can also be done by integrating the corresponding solution from Tada, *et al.*, (1985) with the weight $F(R)/\pi\rho r$.

$$V = 2\frac{1-\upsilon}{\mu}\left[\frac{F(R)R^2}{\rho} + \frac{4}{3}\langle\sigma\rangle R^3\right] \tag{21}$$

where μ and ν are the shear modulus and Poisson's ratio of the material respectively.

4.5 *Dilatancy*

By introducing

$$\lambda = R/R_{cr} \tag{22}$$

and neglecting the weak dependence (18) of F upon R, the law of the mesocrack growth (19), (20) can be expressed as follows

$$\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda} = K_{Ic}\sqrt{\frac{\pi\rho}{2F\langle\sigma\rangle}} \tag{23}$$

Substituting (18) and then (14) into the above equation gives

$$\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda} = \frac{K_{Ic}}{ap}\sqrt{\frac{\pi\rho}{2k_0k_1}} \tag{24}$$

where

$$k_0(R) = \frac{(2-\upsilon)\Psi(\upsilon)\sqrt{I(\upsilon)}}{2\pi(1-\upsilon^2)(1+\upsilon)}\sqrt{2\ln\left(\frac{-R}{J(\upsilon)a}\right)}\sqrt{w},\ k_1 = \frac{2}{5\pi^2}\frac{(2-\upsilon)(3+5\upsilon)}{2\pi(1-\upsilon^2)(1+\upsilon)} \tag{25}$$

$w = Na^3$ is the concentration of initial cracks, N is the number of cracks per unit volume.
Since $\lambda = 1$ when $p = p_{cr}$,

$$p_{cr} = \frac{Ka_{Ic}}{2}\sqrt{\frac{\pi\rho}{2k_0k_1}} \tag{26}$$

The appropriate solution of (24) can then be expressed in the form

$$\lambda = \left(\frac{p_{cr}}{p}\right)^2\left[1-\sqrt{1-\left(\frac{p}{p_{cr}}\right)^2}\right]^2 \tag{27}$$

Here the load of the beginning of unstable mesocrack propagation, p_{cr}, can be interpreted as the uniaxial compressive strength.
Using (21) and taking into account that the inelastic part of the radial strain related to the mesocrack opening (dilatancy, e.g., Brace *et al.*, 1966) is equal to $\Delta\varepsilon_r = MV$, where V is the volume of mesocrack opening (because the mesocracks are oriented parallel to the direction of load, their opening solely contributes to the radial strain, ε_r) and M is the number of mesocracks per unit volume, one has

$$\frac{\Delta_r}{\Delta\varepsilon_r^{\max}} = \frac{p\lambda^2(3+2\lambda)}{5p_{cr}} \tag{28}$$

where

$$\Delta\varepsilon_r^{\max} = \frac{5(1-\upsilon)}{6\mu}\frac{Mk_0^3a^6}{\rho^3k_1^2}p_{cr} \tag{29}$$

It can be observed that in its dimensionless form (27), (28) the relationship between dilatancy and the magnitude of compressive load is universal; it does not depend on parameters k_0 and k_1 reflecting the particular type of the random stress field generators. Hence, it is possible to verify the formulae (27), (28) directly against experimental data without referring to a particular model of wing cracks. For this purpose experiments by Sano *et al.* (1981) on four cylindrical samples of Oshima granite of the height of 11.5 cm and diameter 4.425 cm will be used.
Dilatancy, i.e. the additional volumetric strain (or in this case, the additional radial strain, since the dependencies axial strain vs. load were almost linear) produced by the opening of the mesocracks is determined as the difference between the full measured radial strain and the radial strain extrapolated from the preceding region of linear deformation (region II according to Brace *et al.*, 1966). The end of this region, ie the stress of the beginning of dilatancy, σ_d, has been used as the only matching parameter. Also, for Sample 514, Figure 9, the final (before failure) value of radial strain was unrealistically high, which probably indicated a possibility of an experimental error. Therefore for this sample the maximum value of dilatancy, $\varepsilon_r^{\max}$ was used as another matching parameter. This explains much better correspondence between experimental and theoretical data for Sample 514. Parts (a) in Figures 6-9 show the extrapolation of the region of linear deformation, parts (b) show comparison between the formulae (27), (28) (solid lines) and the normalised experimental data (data points). It is seen that the correspondence is quite good and holds for a wide range of loading rates. Moreover, the determined stresses of the beginning of dilatancy, σ_d, vary only by about 10% for different samples.
A remark should be made here. A previously proposed model (Dyskin *et al.*, 1991; Germanovich *et al.*, 1993) based on the assumption of extensive wing crack growth which was not confirmed by subsequent experiments (eg, Dyskin *et al.*, 1994a, b) showed good correspondence to the experimental data only after involving the second matching parameter, $\varepsilon_r^{\max}$, for all samples (the indication on this, second matching parameter was erroneously missing from the paper by Germanovich *et al.*, 1994a). The comparison between the new model and the previous one is presented by (Dyskin, 1997; the formula for dilatancy proposed there contains a misprint; the plot is correct).

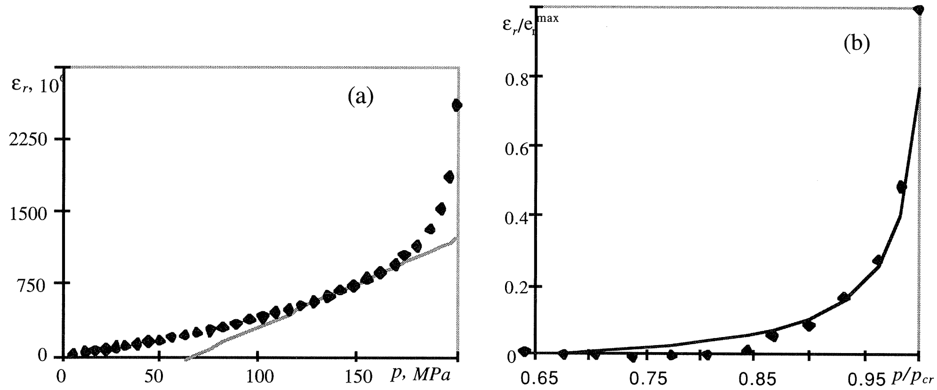

Fig. 6. *Experiments by Sano et al., (1981) on Oshima granite for Sample 511 (loading rate was $4.27 \times 10^{-5}\ s^{-1}$): (a) - extrapolation of Region II; (b) - comparison between experimental (dots) and theoretical (line) data for dilatancy; $\sigma_d = 0.675p_{cr}$.*

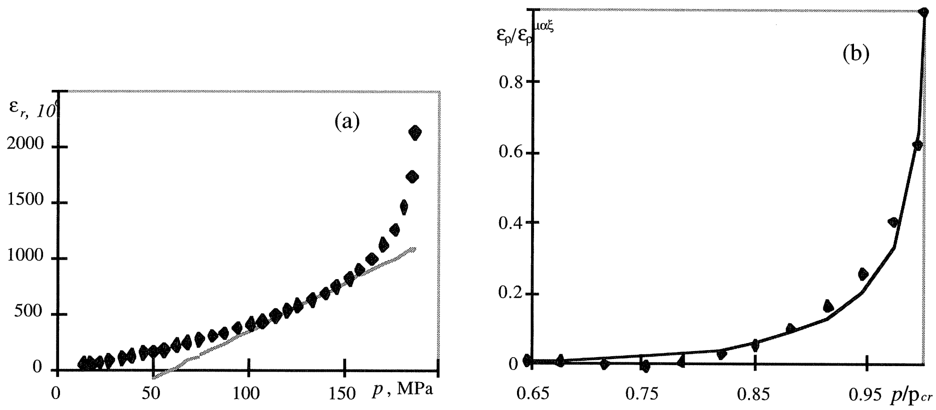

Fig. 7. *Experiments by Sano et al., (1981) on Oshima granite for Sample 512 (loading rate was $4.14 \times 10^{-6}\ s^{-1}$): (a) - extrapolation of Region II; (b) - comparison between experimental (dots) and theoretical (line) data for dilatancy; $\sigma_d = 0.65p_{cr}$.*

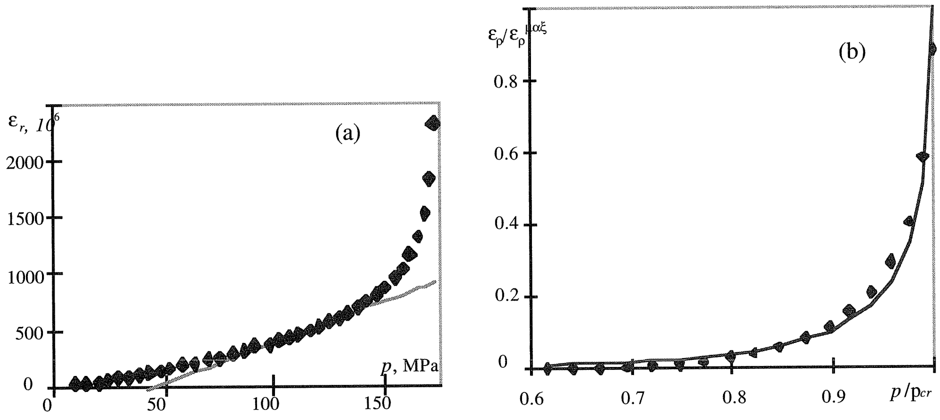

Fig. 8. *Experiments by Sano et al., (1981) on Oshima granite for Sample 513 (loading rate was $3.68 \times 10^{-7}\ s^{-1}$): (a) - extrapolation of Region II; (b) - comparison between experimental (dots) and theoretical (line) data for dilatancy; $\sigma_d = 0.63p_{cr}$.*

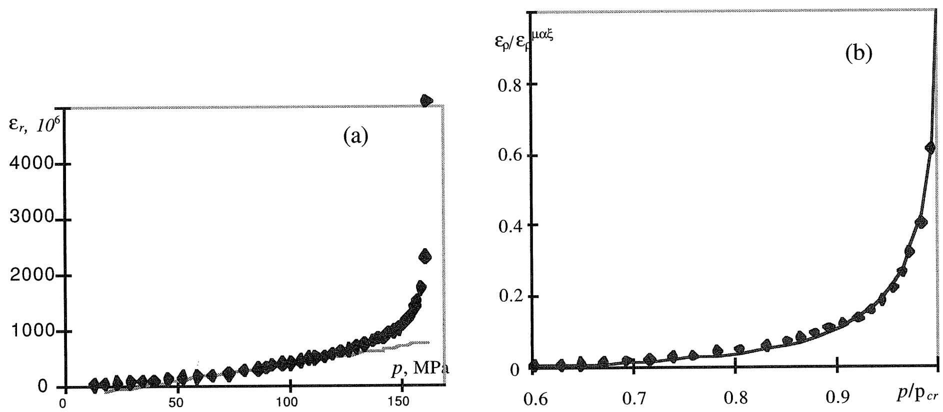

Fig. 9. *Experiments by Sano et al., (1981) on Oshima granite for Sample 514 (loading rate was $3.48 \times 10^{-8}\ s^{-1}$): (a) - extrapolation of Region II; (b) - comparison between experimental (dots) and theoretical (line) data for dilatancy; σ_d=$0.6p_{cr}$.*

4.6 *Stage 3. Failure*

The mesocracks grow stably until the size of the largest mesocrack(s) becomes high enough to enable the background stress to make them grow unstably. This constitutes the beginning of failure. It is not possible at the moment to determine how the unstable phase of mesocrack will proceed or which failure mode will take place (see Dyskin et al., 1996b, for possible mechanisms of splitting and shear failure). Here failure will be associated with the unstable mesocrack propagation, the corresponding load magnitude being taken as the uniaxial compressive strength.

From (19), (26) the criterion of mesocrack growth can be found

$$p\left[\frac{k_0(R)a}{J(\upsilon)\sqrt{\pi\rho}} + 2k_1\sqrt{\frac{R}{\pi}}\right] = K_{\mathrm{Ic}} \tag{30}$$

From here the critical radius, R_{cr} at which the mesocrack starts growing unstably and the corresponding critical load, the uniaxial compressive strength, p_{cr}, are found numerically, Figure 10. Depending on the concentration w of initial cracks the critical radius can exceed the radius of pre-existing crack 10–40 times (Figure 10a) which corresponds to the observations in Chelmsford granite samples (Peng and Johnson, 1972) showing that the initial cracks grow up to 20 times.

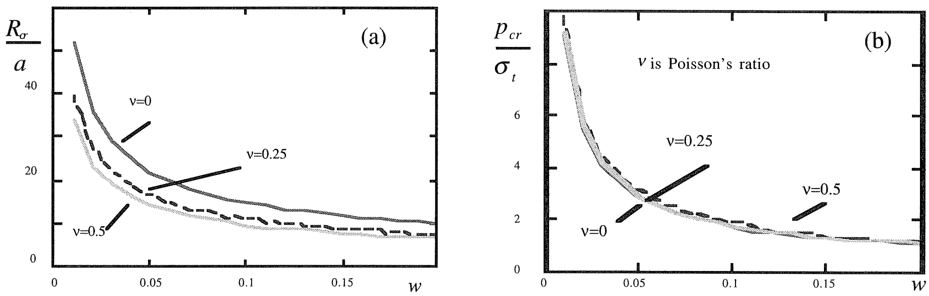

Fig. 10. The critical mesocrack radius (a) and the uniaxial compressive strength (b) vs. the concentration of initial cracks.

The compressive strength shown in Figure 10b is normalised by macroscopic tensile strength determined by unstable growth of a disk-like crack of radius a oriented perpendicularly to the tensile load $\sigma_t = (1/2)K_{\mathrm{Ic}}(\pi/a)^{1/2}$. It is seen that for low concentrations of wing cracks the compressive strength may exceed the tensile one by an order of magnitude and is generally within the range (3–10) which is typical for brittle materials (e.g., Paul, 1968).

5. Conclusions

Failure of heterogeneous materials in uniaxial compression is a three stage process in which the formation and growth of wing cracks is only the first stage. The wing cracks cannot grow extensively to cause failure, so their role is in inducing additional self-equilibrating random stress field. Its normal components acting in the directions perpendicular to the compression direction produce, at the places where they are tensile, new cracks, so-called mesocracks. This starts the second stage of the failure process.

As a mesocrack grows it tends to deviate its path to meet the regions with higher possible local tension. As a result, it grows further under the action of concentrated forces distributed randomly and uniformly with respect to the mesocrack radius. The mesocrack also tends to avoid the wing cracks such that the stress field driving the crack is the stress outside the excluded volumes surrounding the wing cracks. This outside stress has a positive (tensile) mean, which turns the stable mesocrack growth into an unstable one and causes failure.

The growth and opening of mesocracks result in a specific dependence between dilatancy, i.e. inelastic increase of the sample volume, and the applied compressive stress. This dependence has a universal nature independent of the particular model of wing cracks. It corresponds well to the data of uniaxial compressive tests on 4 samples of Oshima granite (Sano *et al.*, 1981) despite markedly different loading rates and resulted strengths.

When mesocracks become large, the background tensile stress makes their growth unstable, which constitutes the last stage, ultimate failure. The load that corresponds to the unstable mesocrack growth can be identified with the uniaxial compressive strength. It is shown that for low concentrations of wing cracks the compressive strength may exceed the tensile one by an order of magnitude and is generally within the range (3-10) typical for brittle materials.

Acknowledgement

The author appreciates the invitation by J.G.M. van Mier to write this paper and acknowledges the support of the Australian Research Council Grant A89801617 (1998–2000).

6 Literature

ADAMS, M. and SINES, G., (1978), Crack extension from flaws in a brittle material subjected to compression. *Tectonophysics*, 49: 97-118.

ALLEGRE, C. J., LE MOUEL, J. L. & PROVOST, A. (1982), Scaling rules in rock fracture and possible implication for earthquake prediction. *Nature* 297: 47-49.

ASHBY, M.F. and HALLAM, S.D. (1986), The failure of brittle solids containing small cracks under compressive stress states. *Acta Metallurgica*, 34: 497-510.

ASHBY, M.F. and SAMMIS, C.G., (1990), The damage mechanics of brittle solids in compression. *Pure and Applied Geophysics*, 133, No. 3: 489-521.

ATKINSON, B. (1981), Cracks in rocks under stress. *Nature* 290: 632.

BAUD, P., REUSCHELE, T. and CHARLEZ, P. (1996). An improved wing crack model for deformation and failure of rock in compression. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 33, N0. 5: 539-542.

BAZANT, Z.P. and XIANG, Y. (1997), Size effect in compression fracture: Splitting crack band propagation. *J. Engineering Mechanics*, 123, No. 2: 162-173.

BAZANT, Z.P., TABARRA, M.R. KAZEMI, T. and PIJAUDIER-CABOT, G. (1990), Random particle model for failure of aggregate or fibre composites. *J. Engineering Mech.*, 116, No. 8: 1686-1705.

BLAIR, S.C., COOK, N.G.W. and DAVELER, S. (1993), A zeroth-order statistical model for fracture in rock in compression. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 30, No. 7: 1619-1622.

BLECHMAN, I. (1997a), Brittle solid under compression. Part I: Gradient mechanisms of microcracking. *Int. J. Solids Struct.*, 34, No. 20: 2563-2581.

BLECHMAN, I. (1997b), Brittle solid under compression. Part II: The problem of macro to micro linkage. *Int. J. Solids Struct.*, 34, No. 20: 2583-2594.

BRACE, W.F. and BOMBOLAKIS, E.G. (1963), A note on brittle crack growth in compression. *J. Geophysical Research*, 68, No. 12: 3709-3713.

BRACE, W.F., PAULDING, B.M., and SCHOLZ, C. (1966), Dilatancy in the fracture of crystalline rocks. *J. Geophysical Research*, 71, No. 16: 3939-3953.

BRADY, B. T. (1976), Laboratory investigation of tilt and seismicity anomalies in rock before failure. *Nature* 260: 108-111.

CANNON, N.P., SCHULSON, E.M., SMITH, T.R., and FROST, H.J. (1990), Wing cracks and brittle compressive fracture. *Acta Metall. Mater.*, 38, No. 10: 1955-1962.

DAVID, H.A. (1970), *Order Statistics*. John Wiley & Sons, Inc. New-York, London.

DEY, T. N and CHI-YUEN WANG (1981), Some mechanisms of microcrack growth and interaction in compressive rock failure. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 18: 199-209.

DU, Y. and AYDIN, A.(1991), Interaction of multiple cracks and formation of echelon crack arrays. *Int. J. for Numer. and Analyt. Methods in Geomechanics*, 15: 205-218.

DYSKIN, A.V. (1989), On the relation between acoustic emission and dilatancy at axial compression of brittle rocks. *Phys. of the Solid Earth*, 25, No. 6: 473-477.

DYSKIN, A.V. (1997), Crack growth under spatially random stress fields. *Advances in Fracture Research, Proc. 9th Intern. Conf. on Fracture*, 4: 2119-2126.

DYSKIN, A.V. and GERMANOVICH, L.N. (1993), Rockburst model based on cracks growing near free surface. *Rockbursts and Seismicity in Mines 93*. R.P. Young (ed.). Proceedings of 3rd International Symposium, Kingston, Ontario, Canada: 169-173.

DYSKIN, A.V. and MÜHLHAUS, H.-B. (1995), Equilibrium bifurcations in dipole asymptotic model of periodic crack arrays. In the book: *Continuum Models for Materials with Micro-Structure*, Edited by H.-B. Mühlhaus, John Wiley & Sons Ltd, 1995: 69-104.

DYSKIN, A.V. and SALGANIK, R.L.(1987), Model of dilatancy of brittle materials with cracks under compression. *Mechanics of Solids*, 22, No. 6: 165-173.

DYSKIN, A.V. and SAHOURYEH, E. (1997). Mechanism of dilatancy and fracture of heterogeneous brittle materials in uniaxial compression. *Intern. J. Fracture,* 84, L45-L50.

DYSKIN, A.V., GERMANOVICH L.N. and USTINOV, K.B. (1996a), Modelling of 3-D crack growth and interaction in uniaxial compression. Proc. *1st Australian Congress on Applied Mechanics*, R.H. Grzebieta, A.K. Wang, J. Marco & Y.C. Lam (Eds.), 1: 139-144. The Institution of Engineers, Australia Preprint of Papers (National Conference Publication No. 96/01).

DYSKIN, A.V., GERMANOVICH, L.N. and SALGANIK, R.L. (1991), A mechanism of deformation and fracture of brittle rocks. In: *Rock Mechanics as a Multidisciplinary Science,* J.-C. Roegiers (ed.). Balkema, Rotterdam, Brookfield: 181 - 190.

DYSKIN, A.V., GERMANOVICH, L.N., and USTINOV, K.B. (1992), On the pore-based mechanism of dilatancy and fracture of rocks under compression. *Proc. 33rd U.S. Symposium on Rock Mechanics*, Santa Fe: 797-806.

DYSKIN, A.V., GERMANOVICH, L.N. and USTINOV, K.B. (1993), Asymptotic solution for long cracks emanated from a pore in compression. *Intern. J. Fract.*, 62: 307-324.

DYSKIN, A.V., GERMANOVICH, L.N., JEWELL, R.J., JOER, H., KRASINSKI, J.S. LEE, K.K., ROEGIERS, J.-C., SAHOURYEH, E. and USTINOV, K.B. (1994b), Study of 3-D mechanisms of crack growth and interaction in uniaxial compression. *ISRM News Journal*, 2, No. 1: 17-20, 22-24.

Dyskin, A.V., Jewell, R.J., Joer, H., Sahouryeh, E. and Ustinov, K.B. (1994a), Experiments on 3-D crack growth in uniaxial compression. *Intern. journal of. Fracture* 65, R77-R83.

Dyskin, A.V., Sahouryeh, E. & Ustinov, K.B. (1996b), The role of micro-plasticity in the mechanisms of shear fracture and splitting in uniaxial compression. *Rock Mechanics Tools and Techniques, Proc. of the Second North American Rock Mechanics Symposium,* M. Aubertin, F. Hassani and H. Mitri (Eds.), 2: 1611-1618.

Fairhurst, C. and Cook N.G.W. (1966), The phenomenon of rock splitting parallel to the direction of maximum compression in the neighborhood of a surface. *Proc. First Congress International Society for Rocks Mechanics*, Lisbon: 687-692.

Fanella, D. and Krajcinovic, D. (1988), A micromechanical model for concrete in compression. *Engineering Fracture Mechanics*, 29, No. 1: 49-66.

Feller, W. (1971), *An Introduction to Probability Theory and its Applications*. Vol. II. John Wiley & Sons, Inc. New-York, London, Sydney, Toronto.

Galybin, A.N. (1985), Formations of cracks on compressing an unbounded brittle body with a circular opening. *J. Applied Mathematics and Mechanics*, 49: 797-799.

Germanovich, L.N. and Dyskin, A.V. (1994c), Virial expansions in problems of effective characteristics. Part I. General concepts. *J. Mech.s Composite Materials*, 30, No. 2: 222-237.

Germanovich, L.N. and Dyskin, A.V. (1988), A model of brittle failure for materials with cracks in uniaxial loading. *Mechanics of Solids*, 23, No. 2: 111-123.

Germanovich, L.N., Dyskin, A.V., and Tsyrulnikov, M.N. (1990), Mechanism of dilatancy and columnar failure of brittle rocks under uniaxial compression. *Transactions (Doklady) of the USSR Academy of Sciences, Earth Science Sections*, 313, No. 4: 6-10.

Germanovich, L.N., Dyskin, A.V., and Tsyrulnikov, M.N. (1993), A model of brittle rock deformations under compression. *Mechanics of Solids*, 28, No. 1: 116-128.

Germanovich, L.N., Lee, K.K, and Roegiers, J.-C. (1994b), *Study of three-dimensional crack initiation and propagation in transparent brittle material subject to compression. Part III. Experimental results on uniaxial compression of frozen PMMA samples*. Rock Mechanics Research Center at The University of Oklahoma: Report RMRC-94-15.

Germanovich, L.N., Salganik, R.L., Dyskin A.V. and Lee, K.K. (1994a), Mechanisms of brittle fracture of rock with multiple pre-existing cracks in compression. *Pure and Applied Geophysics (PAGEOPH)*, 143 1/2/3: 117-149.

Gol'dstein R.V. and Kaptsov, A.V. (1982), The formation of structures of fracture by low-interacting cracks. *Mechanics of Solids*, 17: 157-166.

Gol'dstein, R.V., Ladygin, V.M., and Osipenko, N.M. (1974), A model of the fracture of a slightly porous material under compression or tension. *Soviet Mining Sciences*, 10: 1-9.

Gramberg, J. (1989), *A Non-Conventional View on Rock Mechanics and Fracture Mechanics*. Balkema.

Griffith, A.A. (1924), The theory of rupture. *Proc. 1st Int. Congr. Applied Mech.*, Delft: 55-63.

Hoek, E. and Bieniawski, Z.T. (1965), Brittle fracture propagation in rock under compression. *Intern. J. Fracture*, 1: 137-155.

Horii H. and Nemat-Nasser, S. (1985), Compression-induced microcrack growth in brittle solids: axial splitting and shear failure, *J. Geophys. Research.* 90: 3105-3125.

Horii, H. (1993), Micromechanics and localisation in concrete and rock. *Fracture and Damage of Concrete and Rock*. H.P. Rossmanith (ed.) E & FN Spon, London: 54-65.

Horii, H. and Nemat-Nasser, S. (1986), Brittle failure in compression: splitting, faulting, and brittle-ductile transition. *Phil. Trans. R. Soc. Lond.*, A 319: 337-374.

Isida, M. and Nemat-Nasser, S. (1987), A unified analysis of various problems relating to circular holes with edge cracks. *Engineering Fracture Mechanics*, 27, No. 5: 571-591.

Kemeny, J.M. (1991), A model for non-linear rock deformation under compression due to subcritical crack growth. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 28, No. 6: 459-467.

Kemeny, J. and Cook, N.G.W. (1991), *Micromechanics of Deformation in Rock. Toughening Mechanism in Quasi-Brittle Materials*. Kluwer Academic Publishers, Netherlands.

Krajcinovic, D. (1996), *Damage mechanics*. North-Holland Series in Applied Mathematics and Mechanics. J.D. Achenbach, B. Budiansky, H.A. Lauwerier, P.G. Saffman, L. van Wijngaarden and J.R. Willis (Eds.). Elsevier, Amsterdam, Lausianne, New York, Oxford, Shannon, Tokio.

Labuz, J.F., Dai, S.-T. and Papamichos, E. (1996), Plane-strain compression of rock-like materials. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 33, No. 6: 573-584.

Li, C. and Nordlung, E. (1993) Deformation of brittle rocks under compression – with particular reference to microcracks. *Mechanics of Materials*, 15: 223-239.

Lockner, D.A., Moore, D.E., and Reches, Z. (1992), Microcrack interaction leading to shear fracture. In: *Rock Mechanics*. J.R. Tillerson and W.R. Wawersik (eds.), Balkema: 807-816.

Melin, S. (1983), Why do cracks avoid each other? *Int. J. Fracture*, 23, No. 1: 37-45.

Nemat-Nasser, S. and Horii, H. (1982), Compression-induced nonplanar crack extension with application to splitting, exfoliation, and rockburst. *J. Geophys. Res.*, 87, No. B8: 6805-6821.

Okui, Y and Horii, H. (1994), A micromechanics-based continuum theory for microcracking localization of rocks under compression. *Continuum Models for Materials with Micro-Structure*, Edited by H.-B. Mühlhaus, John Wiley & Sons Ltd, 1995: 27-68.

Papanastasiou, P.C. and Vardoulakis, I.G. (1994). Numerical analysis of borehole stability problem. Soil Structure Interaction: Numerical Analysis and Modelling. J.W. Bull (Ed.) E&FN Spon, Chapter 19, 673-711.

Paul, B. (1968), Macroscopic criteria for plastic flow and brittle fracture. In: *Fracture. An Advanced Treatise* (H. Liebowitz, ed.) II: 313-496.

Peng, S. and Johnson, A.M. (1972), Crack growth and faulting in cylindrical specimens of Chelmsford granite. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr*. 9, 37-86.

Sadowski, T. (1994), Modelling of semi-brittle MgO ceramic behaviour under compression. *Mechanics of Materials*, 18: 1-16.

Sahouryeh, E. and Dyskin, A.V. (1997), The mechanism of 3-D crack growth in uniaxial compression: the role of non-singular stresses. *Int. J. Rock Mech. Min. Sci. and Geomech. Abstr.*, 34, 3-4, Paper No. 071.

Sammis, C.G. and Ashby, M.F. (1986), The failure of brittle porous solids under compressive stress states. *Acta Metallurgica*, 34, No. 3: 511-526.

Sangha, C.M., Talbot, C.J., and Dhir, R.K. (1974), Microfracturing of a sandstone in uniaxial compression. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 11, No. 3: 107-113.

Sano, O., Ito, I., and Takeda, M. (1981), Influence of strain rate on dilatancy and strength of Oshima granite under uniaxial compression. *J. Geophys. Res*. 86, No. B10: 9299-9311.

Santiago, S.D. and Hildsorf, H. (1973), Fracture mechanisms of concrete under compressive loads. *Cement and Concrete Research*, 3: 363-388.

SCHOLZ, C.H., (1968), Microfracturing and the inelastic deformation of rock in compression. *J. Geophys. Res.* 73: 1417-1432.

SCHULSON, E.M., KUEHN, G.A., JONES, D.A., & FIFOLT, D.A. (1991), The growth of wing cracks and the brittle compressive failure of ice. *Acta Metall. Mater.* 39: 2651-2655.

TADA, H., PARIS, P.C., and IRWIN, G.R. (1985), *The stress analysis of cracks. Handbook*. Second edition. Volume II. Paris Productions, Inc. and Del Research Corporation. St. Louis, Missouri.

TALONOV, A.V. and TULINOV, B.M. (1989), Calculation of strains for brittle materials taking into account limiting failure. *J. Appl. Mech. Tech. Phys.*, 30: 470-476.

THEOCARIS, P.S. and SAKELLARIOU, M. (1991) Crack propagation in brittle materials under compressive stresses studied by caustics. *J. Materials Science*, 26: 1640-1646.

VAN MIER, J.G.M. (1992), Scaling in tensile and compressive fracture of concrete. *Applications of Fracture Mechanics to Reinforced Concrete*, A. Carpinteri (Ed.). Elsevier: 95-135.

VAN VLIET, M.R.A. and VAN MIER, J.G.M. (1996), Experimental investigation of concrete fracture under uniaxial compression. *Mechanics of Cohesive-Frictional Materials*, 1: 115-127.

VARDOULAKIS, I. and SULEM, J. (1995), *Bifurcation Analysis in Geomechanics*. Blackie Academic and Professional.

WANG, E.Z. and SHRIVE, N.G. (1995), Brittle fracture in compression: mechanisms, models and criteria. *Engineering Fracture Mechanics*, 52, No. 6: 1107-1126.

WITTMANN, F.H. (1981), Mechanisms and mechanics of fracture of concrete. Advances in Fracture Research (Fracture 81), D. Francos (Ed.): 1467-1487.

WONG, T.-F. (1982), Micromechanics of faulting in Westerly granite. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 19: 49-63.

ZAITSEV, Y. (1983), Crack propagation in a composite material. *Fracture Mechanics of Concrete*: 251-299.