Finite element models for the steady state analysis of moving loads

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The analysis of structures subjected to fast moving loads is a subject of growing interest in railway and pavement engineering. The applications of transient analyses using finite element models, however, are still very limited. The faster a load moves the more elements we need to model the structure. Even at fast workstations and main frame computers a moderate accurate analysis requires a huge amount of computer time.

Many problems can be solved more efficient by application of a steady state analysis using a moving reference system. Based upon this formulation we will develop finite element models that travel together with the moving loads. Such an analysis can be performed with the computer power and execution time necessary for the solution of a common static problem, thus at a normal PC. Especially in the design phase such an analysis is very attractive.

Key words: **moving loads, finite elements, railway engineering**

1 Introduction

In the underlying report we model the moving load problem by a 2-D plane stress/strain model with moving loads at the edge of the model -see figure 1-. These loads move with a constant speed c along the x-axis. With respect to the y-axis (the depth) we take layers (and elements) with increasing thickness B. The top layer supports the rail, the bottom layer is either just a layer or a layer of springs and dashpots only.

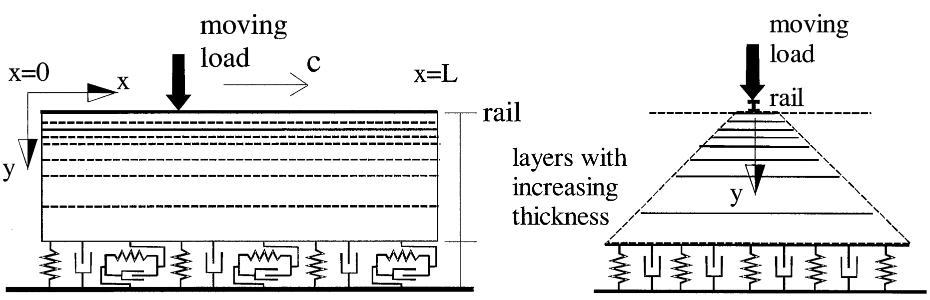

Fig. 1.

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The analysis will be performed following the theory of linear elasticity; nonlinear properties are ignored. Because of the time dependent character of the problem we have to consider inertia properties and damping properties; the faster the train moves, the more important the dynamic properties are.
The structural components used here are a Timoshenko beam at the top of the half-plane (the rail) and a series of elastic layers. The thickness and the properties of the layers may vary with the depth. Energy dissipation is realised by damping properties of the layers and the surroundings.
The response to the moving loads is calculated by a steady state analysis. Finite element mesh and results are defined with respect to a moving reference system that travels together with the loads. Discretisation and analysis are carried out following the common procedures of f.e.m. techniques.

2 Equations of motion

Introducing the stress vector σ, body loads $\boldsymbol{b}$, displacement field $\boldsymbol{u}$, differential operator L and density matrix $\boldsymbol{R}$ we denote the equations of motion by

$$L\sigma^{\mathrm{e}} + \boldsymbol{b}^{\mathrm{e}} = \boldsymbol{R}\ddot{\boldsymbol{u}}^{\mathrm{e}} \tag{1}$$

where

$$L = \begin{pmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{pmatrix} \qquad \boldsymbol{R} = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_{\mathrm{xx}} \\ \sigma_{\mathrm{yy}} \\ \sigma_{\mathrm{xy}} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_{\mathrm{x}} \\ b_{\mathrm{y}} \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} u_{\mathrm{x}} \\ u_{\mathrm{y}} \end{bmatrix}$$

Assuming a load moving with constant speed c in the x-direction we can formulate the steady state solution of the equations of motion by

$$\boldsymbol{u}(\bar{x}, y, t) = \boldsymbol{u}(\bar{x} - ct, y)$$

Introducing $x = \bar{x} - ct$ we may substitute

$$\partial/\partial\bar{x} = \partial/\partial x$$
$$\partial/\partial t = -c\,\partial/\partial x$$

where $\bar{x}$ is fixed and x travels together with the loads.
Substitution hereof in (1) yields

$$L\sigma^{\mathrm{e}} + \boldsymbol{b}^{\mathrm{e}} - c^2\boldsymbol{R}\boldsymbol{u}^{\mathrm{e}}_{,\mathrm{xx}} = 0 \tag{2}$$

Condition (2) has to hold everywhere within the moving mesh, or as we will say, the moving elements.

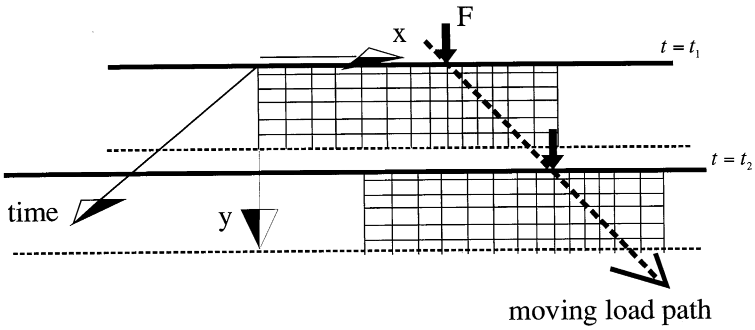


Fig. 2.

3 Constitutive equations

The properties of the layers are modelled following the theory of linear elasticity. Under static load conditions the stress strain relations are given by

$$\sigma = D\varepsilon$$

Under dynamic load conditions energy dissipation is a very important issue for the modelling of soil structures. We will consider two ways of energy dissipation, namely energy dissipation in the structure body by material damping and energy dissipation by energy flow into the surroundings and the bottom.

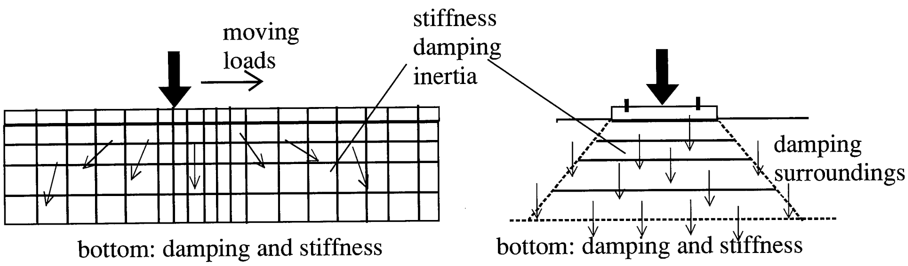


Fig. 3. Energy dissipation.

Material damping is introduced by the addition of a contribution of strain velocities to the stress strain relations following

$$\sigma = D\varepsilon + \tau D\dot{\varepsilon}$$

where τ is the material damping parameter.

For a steady state analysis we can write

$$\sigma = D\varepsilon + \tau c D \varepsilon_{,x}$$

The constitutive equations are given by

$$\begin{aligned} \sigma_{xx} &= s_{11}\varepsilon_{xx} + s_{12}\varepsilon_{yy} - \tau c s_{11} u_{x,x} - \tau c s_{12} u_{y,yx} \\ \sigma_{yy} &= s_{12}\varepsilon_{xx} + s_{22}\varepsilon_{yy} - \tau c s_{12} u_{x,x} - \tau c s_{22} u_{y,yx} \\ \sigma_{xy} &= s_{33}\gamma_{xy} - \tau c s_{33} u_{x,yx} - \tau c s_{33} u_{y,xx} \end{aligned} \qquad (3)$$

4 Boundary conditions

It is dependent on the material properties whether the loads travel slower or faster then the critical speeds. For layers with critical speeds larger then the velocity of the moving loads we apply, at sufficient distance from the load application points, kinematic boundary conditions at $x = 0$ and $x = L$. For layers with critical speeds smaller then the velocity of the moving loads we apply initial conditions at $x = L$. Usually we take the compression wave velocity as the critical speed.

A most complicated boundary condition is the model of the energy dissipation by the surroundings.

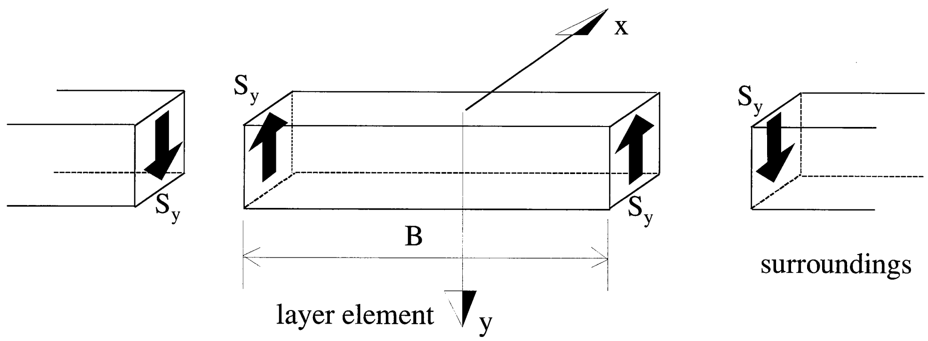


Fig.4. Damping forces at contact face between track and surroundings.

The transfer of impulses transverse to the track structure occurs primarily by shear damping forces S_x andS_y. These damping forces, introduced to simulate a silent surrounding, are given by

$$\begin{aligned} S_x &= \sqrt{G\rho}\, u_x = -c\sqrt{G\rho}\, u_{x,x} \\ S_y &= \sqrt{G\rho}\, \dot{u}_y = -c\sqrt{G\rho}\, u_{y,x} \end{aligned} \qquad (4)$$

Because of the weak knowledge of the damping properties we introduce per layer a factor ζ to be multiplied with the damping coefficient. We will write

$$s = -c C_{sur} u_{,x}$$

where

$$s = \begin{bmatrix} S_x \\ S_y \end{bmatrix} \qquad \text{and} \qquad C_{sur} = \zeta \begin{bmatrix} \sqrt{G\rho} & 0 \\ 0 & \sqrt{G\rho} \end{bmatrix}$$

In our model we will ignore the stiffness properties of the surroundings. Addition of (4) to the equations of motion (2) yields

$$\begin{aligned} B(\sigma_{xx,x} + \sigma_{xy,y} + b_x - \rho c^2 u_{x,xx} - 2S_x) &= 0 \\ B(\sigma_{xy,x} + \sigma_{yy,y} + b_y - \rho c^2 u_{y,xx} - 2S_y) &= 0 \end{aligned} \qquad (5)$$

The parameter ζ has to be input with the layer properties.

5 Finite element model

To develop finite element models we apply Galerkin's variational method to the differential equations of (2) and the boundary conditions. Following Galerkin's method we have to satisfy to the condition that

$$R^e = \iint_A \delta u^{e^T} \{(L\sigma^e + b^e - c^2(Ru^e_{,xx}))B - 2s^e\} dA + \text{boundary contributions} = 0 \qquad (6)$$

for every kinematically admissible variation δu^e.

Application of Green's theorem yields

$$\left(R^e = \iint_A \{(-\delta\varepsilon^{e^T}\sigma^e + \delta u^{e^T}(b^e + c^2 \delta u^{e^T}_{,x} Ru^e_{,x}))B - 2\delta u^{e^T} s^e\}\right) + \text{boundary contributions} = 0$$

We will satisfy Galerkin's variational condition by application of finite element models. The general outline of the finite element method [1],[4] has been based upon assumptions of the displacement field u following

$$u^e = N^e a^e$$

a^e nodal displacements or element degrees of freedom (d.o.f.'s)
N^e interpolation matrix built up with shape functions

Substitution of these assumptions into the Galerkin condition yields per element a system of equations following

$$H^e a^e = f^e$$

The assembly of the element 'stiffness' matrices and the application of the boundary conditions returns the system of equations

$$Ha = f \qquad (7)$$

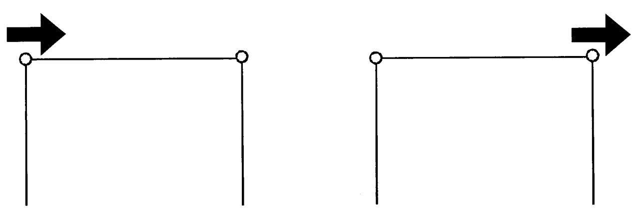
with moving loads **f**.

Because of the damping contributions of the layers and the surroundings we obtain a *nonsymmetric* matrix ***H***.
For our models we applied the 4-node rectangular finite elements based upon a bilinear displacement field [1]. Because of the low order of the shape functions these models cannot take into account the strain velocities $u_{x,xx}$ and $u_{y,xx}$. This may put restrictions on the applicability of the method for stiff layers with notable damping properties.

6 Numerical stability

If the speed of the moving load is subsonic with respect to all layers the system of equations is formulated as a boundary problem. Numerically the associated finite element models are unconditional stable. If, however, the speed of the moving loads is supersonic with respect to one or more layers we have to formulate initial value conditions at $x = L$. These models can show numerical unstable solutions if segment values Δx are taken too large.

Finite elements developed with respect to x-axis and y-axis can be developed in a similar way as time space elements [3] with respect to time axis and y-axis only. Direct integration methods such as the Newmark-β method can be interpreted as nonconforming time space elements. Unconditional numerical stability -thus stability for every value of \$, or \$ in a steady state analysis- is guaranteed if $\beta \geq 0.25$. In [3] it has been shown that \$ corresponds with constant values of the strains with respect to time axis -or x-axis- .
Taking the strains constant with respect to x guarantees unconditional numerical stability for both subsonic and supersonic speeds of the loads.
In practice we take the averaged values of $u^{e}_{x,y}$ and $u^{e}_{y,y}$ for the evaluation of the stiffness contribution of H^e.
A second source of numerically unstable behaviour is associated with the material damping.
A straight on modelling of the material damping shows easily an unstable numerical response, even with moderate low speeds. A simple solution of the problem is to apply backward differences of the strain velocities for the elaboration of the material damping if $c < c_{crlt}$ and forward differences if $c > c_{crlt}$ [2]. In fact we 'lump' the damping forces in a backward or a forward way.



backward lumping of the damping force

forward lumping of the damping force

Fig. 5.

7 Examples

To verify the method a load moving at the edge of a half plane has been analysed. The exact solution [5], a numerical transient analysis [6] and the method outlined here are compared. The material coefficients of the analysed problem are given by $E = 10^8 \mathrm{N/m^2}$, $\nu = 0.2$, $\rho = 2000\ \mathrm{kg/m^3}$ and Rayleigh wave velocity $c_R = 130.65\ \mathrm{m/s}$. No damping is considered.
The finite element analysis is performed using 80 × 25 elements of 1.50 × 1.50 m each – see figure 6 –. Our reference value is the amplification of stress σ_{yy} at $x = 0.75$ m and $y = 5.25$ m (with respect to the load application point) – see figure 7 –.

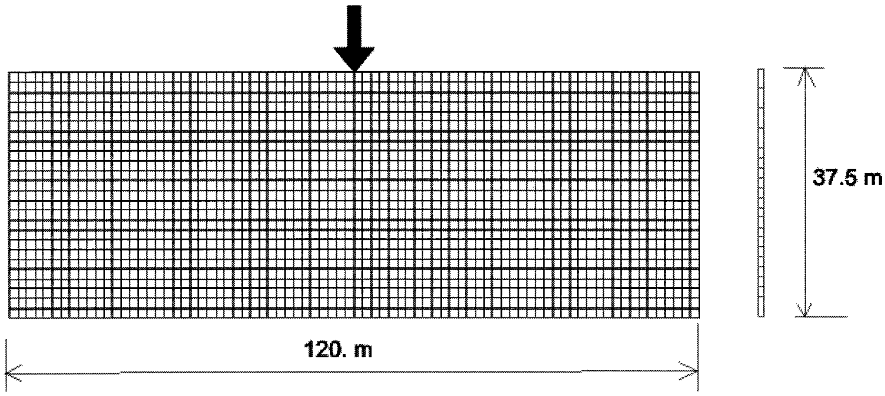


Fig. 6.

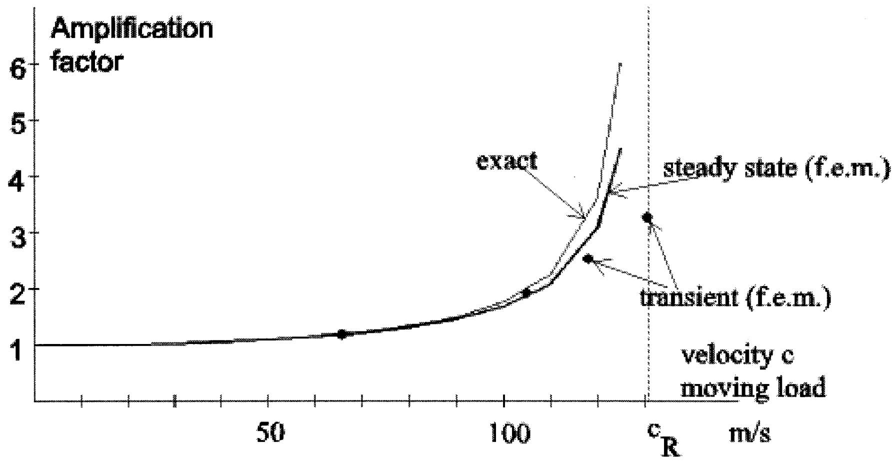


Fig. 7.

In an attempt to improve the model the thickness of the element is increased with growing depth. The material coefficients are taken for a weak soil, in this case $E = 1.5\ 10^8 \mathrm{N/m^2}$, $\nu = 0.4$, $\rho = 1500\ \mathrm{kg/m^3}$ and Rayleigh wave velocity $c_R = 55.29\ \mathrm{m/s}$. Here we consider also the damping contributions, taking $\tau = 0$ and $\zeta = 0$, $\tau = 0.02$ and $\zeta = 0$, $\tau = 0.02$ and $\zeta = 0.2$ respectively.
Finite element model and results are shown in the figures 8 and 9. The reference value is taken at $x = 0.25$ m and $y = 1.25$ m from the load application point.

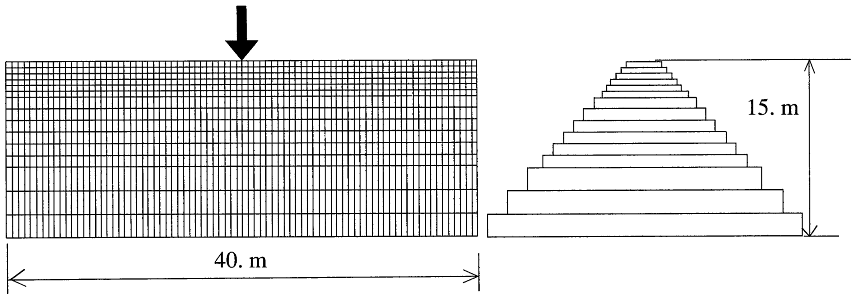


Fig. 8.

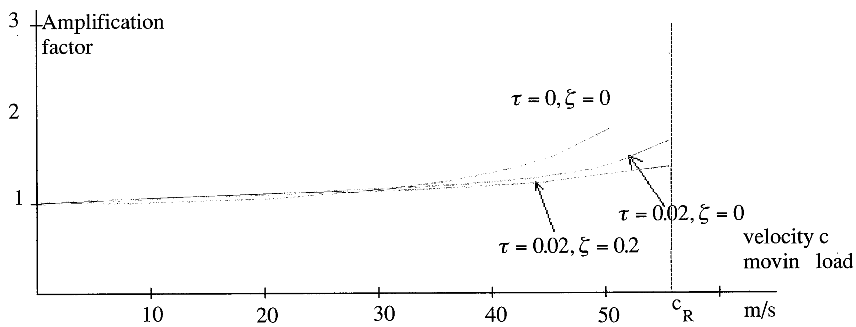
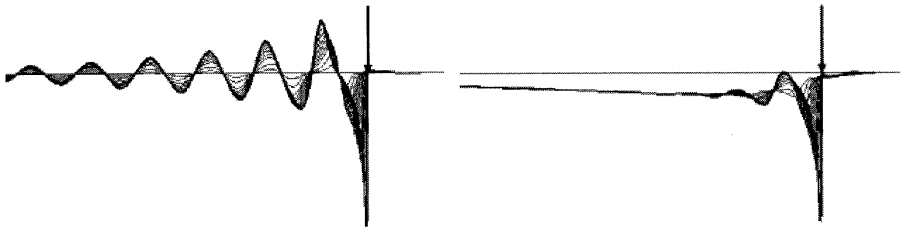


Fig. 9.

The analyses show that both the shape of the cross section and the damping reduce the amplification factor considerably. The analysis of supersonic speeds (c=70 m/s) showed to be dependent on the model of the bottom boundary. We considered a rigid bottom and a silent bottom and added some damping ($\tau = 0.01$ and $\zeta = 0.1$). The results are shown in figure 10.



rigid bottom case *silent bottom case*

Fig. 10. Displacements for c=70 m/s

The explanation of the waves in the rigid bottom case is simple: a rigid boundary reflects the (shear) waves, and a silent boundary does not.

8 Discussion

The method as outlined here shows practical value for the analysis of moving load problems.
The numerical models have shown to be sufficiently stable for the analysis of loads travelling with subsonic and supersonic speeds. Many problems of moving load systems can be analysed with little computer power and modelling efforts.
Some shortcomings, however are noticed too. The analysis results are extremely dependent both on the shape of the cross section and the damping properties. Usually very little is known about the damping properties. The formulation of a realistic 2D model, that requires the modelling of the damping properties of the surroundings, is a very difficult task. For bending problems the neglecting of the material damping in the driving direction of the moving loads may be difficult to justify.

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