A Stoneley-Gibson-Varga elastic stratum

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Non-linear governing equations and some special non-linear static solutions
The influence of geometrical non-linearity on the linear theory of static subgrade reaction and surface wave motion of a Gibson soil.

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1. Introduction

In linear geodynamics the Stoneley model for the transmission of Rayleigh waves in the earth consists of an incompressible, isotropic elastic half-space $X_1 \geq 0$ in which the shear modulus μ increases linearly with increasing depth X_1 according to the equation $\mu = \mu_0 + mX_1$, with μ_0 and m are positive constants, (Stoneley (1934)).
In soil mechanics a Gibson soil is a Stoneley elastic half-space with zero shear modulus at the upper surface $X_1 = 0$, so that $\mu_0 = 0$ and $\mu = mX_1$.
In his famous 1967-paper Gibson showed that the upper surface of this elastic deep stratum reacts under static normal loading like a uniform bed of springs, a so-called Winkler foundation (Gibson (1967)). Any point of the upper-surface $X_1 = 0$ settles an amount $w\,(X_1 = 0)$, directly proportional to the local intensity $q(X_1 = 0)$ of applied normal stress according to the law $w(X_1 = 0) = q(X_1 = 0)\,/\,(2m)$; outside the loaded area the upper-surface does not settle. It has been noticed (Kruijtzer (1976)) that the induced deformation at the locations $X_1 > 0$ is irrotational and the settlement $w(X_1)$ of a point at the level $X_1 > 0$ is directly proportional to the all-round pressure $-p(X_1)$ at that point according to $w(X_1) = p(X_1)\,/\,(2m)$. These settlements at the levels $X_1 > 0$ decrease with increasing horizontal or vertical distances from the loaded surface area.
In the appendix 1 we give some results of the linearised elastodynamics of the Stoneley half-space. One of these results concerns us particularly. When we analyse a group of plane harmonic waves of wavelength $\lambda = 2\pi/\kappa$ travelling through a Gibson half-space in a horizontal direction with velocity c, it is found that there exist irrotational Rayleigh waves with wave velocity $c = ((\rho g + 2m)\,/\,(\rho\kappa))^{1/2}$ in which ρ is the mass density and g is the constant acceleration due to gravity. These irrotational waves are mathematically similar to the (irrotational) gravitational waves in deep water with $c = (g\,/\,\kappa)^{1/2}$.

In this paper we focus our attention to the geometrically non-linear theory of elasticity of a Gibson half-space originally being subjected to a hydrostatic stress distribution.
We choose the constitutive equation of a Varga material, which correlates the actual (Cauchy) stress $\overline{\overline{\sigma}}$ and the so-called left stretch tensor $\overline{\overline{V}}$ (Varga (1966)). In fact, we combine the ideas of Stoneley, Gibson and Varga. Therefore we call our medium a Stoneley-Gibson-Varga stratum.

2. Governing equations

Preliminaries and notations

Let there be given at fixed time $t = 0$ an incompressible, isotropic, unstrained elastic body B which occupies the lower half-space and possesses a volume V, a plane upper boundary S and a boundary S_∞ at infinity. At this time the body is at a state of rest. The density of the body is assumed uniformly distributed in V. The rigidity (the shear modulus) of the body is assumed to increase linearly with depth from zero at the upper boundary S. It is assumed that in the body acts a hydrostatic stress distribution due to a constant gravity. Such a stress distribution does not affect the unstrained state of the body since the body is incompressible and isotropic. By virtue of the conditions of equilibrium this initial all round pressure distribution increases linearly with depth from zero on S.
From a certain instant of time the body is forced to undergo a finite deformation. The body is deformed into the body B' with volume V', upper boundary S' and boundary S_∞' at infinity.
It will appear to be convenient to use the Lagrangian (material) description as well as the Eulerian (spatial) description of the deformation.
The position of a generic particle P of the unstrained body B at a state of rest is identified by the Cartesian co-ordinates X_1, X_2, X_3 or its radius vector $\overline{X} = (X_1, X_2, X_3)$.
The origin O of this right-handed co-ordinate system lies on S, the X_1-axis being directed downwards.
The position of the same particle P but now of the deformed body B' is given by the Cartesian co-ordinates x_1, x_2, x_3 or its radius vector $\overline{x} = (x_1, x_2, x_3)$. The origin o of this right-handed co-ordinate system coincides with O, the x_1-axis directed downwards.

The motion of the material points P is expressed by a one-parameter family of mappings. In the Lagrangian description:

$$\overline{x} = \overline{x}(\overline{X}, t), \quad x_i = x_i(X_1, X_2, X_3, t) \tag{2.1}$$

or, alternatively, in the Eulerian description:

$$\overline{X} = \overline{X}(\overline{x}, t), \quad X_i = X_i(x_1, x_2, x_3, t) \tag{2.2}$$

where function and function values are denoted by the same symbols. The functions are supposed to be sufficiently continuously differentiable as needful with respect to their arguments. In our analysis we follow Malvern (1969).

In the Lagrangian description (2.1) the velocity $\bar{v} = \bar{v}(\bar{X},t)$ and the acceleration $\bar{a} = \bar{a}(\bar{X},t)$ of a particle P are defined by

$$\bar{v} = \bar{v}(\bar{X},t) = \left(\frac{\partial \bar{x}(\bar{X},t)}{\partial t}\right)_{\bar{X}} \equiv \frac{d\bar{x}(\bar{X},t)}{dt} \tag{2.3a}$$

and

$$\bar{a} = \bar{a}(\bar{X},t) = \left(\frac{\partial \bar{v}(\bar{X},t)}{\partial t}\right)_{\bar{X}} \equiv \frac{d^2\bar{x}(\bar{X},t)}{dt^2} \tag{2.3b}$$

where d/dt denotes the material time derivative, i.e., the partial time derivative with $\bar{X}$ held constant.
Substitution of (2.2) in (2.3) yields the Eulerian description of $\bar{v} = \bar{v}(\bar{x},t)$ and $\bar{a} = \bar{a}(\bar{x},t)$ at some point $\bar{x}$, the current position of the points P.
In rectangular Cartesians:

$$v_k = v_k(x_1, x_2, x_3, t) \tag{2.4a}$$

and

$$a_k = a_k(x_1, x_2, x_3, t) \tag{2.4b}$$

with

$$a_k = \frac{\partial v_k}{\partial t} + v_m \frac{\partial v_k}{\partial x_m}, \text{ sum on } m. \tag{2.4c}$$

In (2.4c) $\partial/\partial t$ represents the (local) time derivative with x_m ($m = 1,2,3$) held constant.
In general, whenever the same letter subscript occurs twice in a term, that subscript is to be given all possible values and the results added together.
The components of the material deformation gradient $\bar{\bar{F}}$ and the spatial deformation gradient $\bar{\bar{F}}^{-1}$ are defined by, respectively,

$$d\bar{x} = \bar{\bar{F}} \cdot d\bar{X}\ ,\ \ dx_k = \frac{\partial x_k}{\partial X_m} \cdot dX_m \tag{2.5a}$$

and

$$d\bar{X} = \bar{\bar{F}}^{-1} \cdot d\bar{x},\ \ dX_m = \frac{\partial X_m}{\partial x_k} \cdot dx_k \tag{2.5b}$$

The material deformation gradient $\bar{\bar{F}}$ with elements $\partial x_k/\partial X_m$ and the spatial deformation gradient

$\overline{\overline{F}}^{-1}$ with elements $\partial X_k/\partial x_m$ are related to each other by $\overline{\overline{F}}\cdot\overline{\overline{F}}^{-1} = \overline{\overline{1}}$ or $\overline{\overline{F}}^{-1}\cdot\overline{\overline{F}} = \overline{\overline{1}}$, with $\overline{\overline{1}}$ as the unit tensor. These two relationships are expressed by two sets of nine simultaneous equations,

$$\frac{\partial x_k}{\partial X_m} \cdot \frac{\partial X_m}{\partial x_i} = \delta_{ki} \ , \ \frac{\partial X_k}{\partial x_m} \cdot \frac{\partial x_m}{\partial X_i} = \delta_{ki} \tag{2.5c}$$

where δ_{ki} are the Cartesian components of the unit tensor: $\delta_{ki} = 1$ if $k = i$, $\delta_{ki} = 0$ if $k \neq i$. These relations (2.5c) play an important role in our further analysis. The condition of incompressibility of the body requires that

$$\det(\overline{\overline{F}}) = \det(\overline{\overline{F}}^{-1}) = 1 \tag{2.6a}$$

or, alternatively,

$$\frac{\partial v_k}{\partial x_k} = 0 \tag{2.6b}$$

The polar decomposition theorem (Malvern (1969), p. 178) establishes the unique representation of the material deformation gradient tensor by $\overline{\overline{F}} = \overline{\overline{R}} \cdot \overline{\overline{U}} = \overline{\overline{V}} \cdot \overline{\overline{R}}$, in which $\overline{\overline{R}}$ is a proper orthogonal tensor representing the local rigid rotation of the material, and $\overline{\overline{U}}$ and $\overline{\overline{V}}$ are symmetric tensors representing the pure deformation of the body material. The tensor $\overline{\overline{U}}$ is known as the right stretch tensor. The tensor $\overline{\overline{V}}$ is called the left stretch tensor.
Thus,

$$\overline{\overline{F}} = \overline{\overline{R}} \cdot \overline{\overline{U}} = \overline{\overline{V}} \cdot \overline{\overline{R}} \tag{2.7a}$$

In Cartesian components we may write

$$\frac{\partial x_k}{\partial X_m} = V_{k\ell} \cdot R_{\ell m} = R_{k\ell} \cdot U_{\ell m} \tag{2.7b}$$

Equations of motion

The physical components of the actual stress $\overline{\overline{\sigma}}$ at a point $\overline{x}$ in the deformed body B' (the Cauchy stress, the Eulerian stress or the "true" stress) are denoted by σ_{ik}. The component σ_{ik} represents the component in the direction of x_k of the traction on a material surface in the deformed body B' with a normal into the direction of x_i. The spatial or Eulerian equations of motion are with respect to deformed body

$$\left.\begin{aligned} &\frac{\partial \sigma_{ij}}{\partial x_i} + b_j = \rho a_j \\ &\sigma_{ij} = \sigma_{ji} \end{aligned}\right\} \text{, in } V' \tag{2.8}$$

where $b_1 = \rho g$, $b_2 = 0$, $b_3 = 0$ are the components of the constant body force. Here ρ is the uniform and constant mass density and g is the uniform and constant acceleration due to gravity.

Constitutive equation

In this paper a possibly rather unknown formulation of the stress-strain relationship of an incompressible, isotropic material will be used. In the appendix 2 a formal justification of this relationship is presented. A more or less intuitive derivation of the relationship can be given as follows.

R.S. Rivlin (1948) considered an incompressible, isotropic elastic solid. By use of the concept of the strain ellipsoid Rivlin showed that a natural generalisation of Hooke's law in the case of small strains of such a solid to the case of large strains is given by:

$$\overline{\overline{\sigma}} = -p\overline{\overline{1}} + \mu\overline{\overline{B}} \quad , \overline{\overline{B}} = \overline{\overline{F}} \cdot \overline{\overline{F}}^T = \overline{\overline{V}} \cdot \overline{\overline{V}}$$

or (2.9)

$$\sigma_{ij} = -p\delta_{ij} + \mu B_{ij} \quad , B_{ij} = V_{ik} \cdot V_{kj}$$

with $-p$ an all round pressure ($p > 0$) and $\mu > 0$ is the "shear modulus", with $\mu = 2E/(1 + \nu)$, in which E is the Young's modulus and ν is the Poisson's ratio; $\overline{\overline{B}}$ is the Finger deformation tensor. Rivlin assumes $\nu = 1/2$, so that $\mu = E/3$. Rivlin describes a material obeying the law (2.9) as an incompressible, neo-Hookean material. Rivlin showed that if a cube of incompressible, neo-Hookean material is subjected to a pure, homogeneous deformation and the stress components in the deformed body are prescribed or specified, the state of strain of the deformed body is uniquely determined. On the other hand, Rivlin showed that if the state of strain is prescribed, as being exemplified in the case of simple shear, the state of stress in the deformed body is not uniquely determined. This type of lack of uniqueness is often met with non-linear elasticity of incompressible materials (Rivlin (1948)).

The principal axes of the Finger deformation tensor $\overline{\overline{B}} = \overline{\overline{F}} \cdot \overline{\overline{F}}^T$ and the principal axes of the symmetric left stretch tensor $\overline{\overline{V}}$ coincide, since with $\overline{\overline{F}} = \overline{\overline{V}} \cdot \overline{\overline{R}}$ and $\overline{\overline{R}}^{-1} = \overline{\overline{R}}^T$ there is the relation $\overline{\overline{B}} = \overline{\overline{V}} \cdot \overline{\overline{R}} \cdot (\overline{\overline{V}} \cdot \overline{\overline{R}})^T = \overline{\overline{V}} \cdot \overline{\overline{R}} \cdot \overline{\overline{R}}^{-1} \cdot \overline{\overline{V}} = \overline{\overline{V}}^2$. Let λ^2 be a (positive) principal value of $\overline{\overline{B}}$ and λ the corresponding positive principal value of $\overline{\overline{V}}$. Then in the case of small strains the quadratic strain $(\lambda^2 - 1)$ and the linear strain $(\lambda - 1)$ possess scantly different values. In view of this fact we replace the constitutive equation by the stress-strain relationship

$$\overline{\overline{\sigma}} = -p\overline{\overline{1}} + 2\mu\overline{\overline{V}}$$

or (2.10)

$$\sigma_{ij} = -p\delta_{ij} + 2\mu V_{ij}$$

A material that obeys this constitutive relationship is called sometimes a Varga material (Spencer (1982), Varga (1966)).
The use of $\overline{\overline{V}}$ as measure of the strain is seldom practical since the elements of $\overline{\overline{V}}$ are in general not rational functions of the displacement gradients. However, there is no physical objection against the use of $\overline{\overline{V}}$ as measure of the strains.
In this study it is supposed that the shear modulus varies within the undistorted configuration according to the law

$$\mu = mX_1 \, , \; X_1 \geq 0 \tag{2.11}$$

where m is a positive constant.

Comment
In soil mechanics the deformations of strata are often determined by regarding these strata as being elastic provided that appropriate values of the elastic parameters are selected. In general the shear rigidity of soils is assumed to be a functional depending on, for example, the average effective confining pressure, the ambient stress history and vibration history, the void ratio and other geometrical and physical soil characteristics (Richart et al. (1970) p. 152).
In the cases of normally consolidated clays and sands the main feature of this functional dependency is that the shear modulus varies with the square root of the isotropic effective stress (Richart et al. (1970) p. 353). The constitutive equations (2.10) and (2.11) do not satisfy this main feature since the shear modulus is only a linear function of the initial isotropic stress and does not depend on an additional isotropic stress due to additional loadings such as surface loadings. However, the only aim of this study is to investigate the effect of geometrical non-linearity on the results of the linear theories of static subgrade reaction and surface wave motion of a Gibson half-space.

3. Pure strain (no rigid rotation)

Dynamics
The search for solutions in finite elasticity often meets with insurmountable mathematical difficulties, but appropriate assumptions can be made for the determination of exact solutions. In this study it is assumed that there exists a non-empty class of non-trivial boundary value problems which involve a pure strain in the body. Of course, this assumption of pure strain cannot be anticipated on physical grounds.
The condition of pure strain implies that the rotation tensor reduces to the unit tensor, so that the left stretch tensor equals the material deformation gradient tensor, which then becomes symmetric.
Under these circumstances the constitutive equations (2.10) reduce to

$$\sigma_{ij} = -p\delta_{ij} + 2\mu \frac{\partial x_i}{\partial X_k} \delta_{kj} \, , \; \mu = mX_1 \tag{3.1a}$$

or, alternatively,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu \text{ cofactor } \left(\frac{\partial X_k}{\partial x_i}\right) \delta_{kj}\,,\ \mu = mX_1 \tag{3.1b}$$

since $\det(\partial x_i / \partial X_j) = 1$. With $\mu = mX_1$, the elimination of σ_{ij} between the constitutive equations (3.1.) and the equations of motion (2.8), and the use of the condition of incompressibility yield the following equations of motion:

$$-\frac{\partial p}{\partial x_1} + (\rho g + 2m) = \rho a_1$$

$$-\frac{\partial p}{\partial x_2} = \rho a_2 \tag{3.2}$$

$$-\frac{\partial p}{\partial x_3} = \rho a_3$$

The derivation of these equations requires some algebra, which is presented in appendix 3.
The resulting Eulerian equations of motion (3.2.) are mathematically similar with those governing the irrotational motion of an incompressible perfect fluid. Of course, in contrast with a perfect fluid, the body under consideration can support shearing stress. Thus the mathematical analogy applies only to the equations of motion and the condition of incompressibility. On account of this rather surprising result there exists a single-valued acceleration potential Ω (say) (Malvern (1969), p. 455, Lamb (1945), p. 19) which is given by

$$\Omega = \Omega(\bar{x}, t) = \frac{1}{\rho}\left\{-p + (2m + \rho g)x_1 + \psi(t)\right\} \tag{3.3a}$$

where $\psi(t)$ is a function of t only, to be determined from the boundary conditions. With (3.2) and (3.3a) it follows that

$$a_i = \frac{\partial \Omega}{\partial x_i} \tag{3.3b}$$

This equation states that the motion is circulation-preserving, with a single acceleration potential (compare Kelvin's theorem on the conservation of circulation in a barotropic perfect fluid under conservative body forces, (Malvern (1969), p. 434). We notice that our analysis shows that the condition of pure strain imposed on the deformation of the body under consideration, forces all motions to satisfy Truesdell's theory of quasi-equilibrated motions, (Truesdell (1965), page 208). According to this theory $\bar{x} = \bar{x}(\bar{X}, t)$ defines a static deformation for each value of the time t, and the shear stresses in the motion are the same as those that correspond to the equilibrium in the configuration occupied by the body at the time t.

Statics

In the case of static deformations the integrated conditions of equilibrium (3.2) (with $a_i = 0$) lead to the equation

$$-p + (\rho g + 2m)x_1 = C,\ X_1 \geq 0 \tag{3.4}$$

where C is the constant of integration to be determined from the boundary conditions (compare (3.3a)). In the undeformed state there is the hydrostatic pressure distribution

$$\sigma_{x_1x_1} = \sigma_{x_2x_2} = \sigma_{x_3x_3} = -\rho g X_1,\ X_1 \geq 0 \tag{3.5a}$$

Under these circumstances the constitutive equations (3.1) reduce to

$$\sigma_{x_1x_1} = \sigma_{x_2x_2} = \sigma_{x_3x_3} = -p + 2mX_1 \tag{3.5b}$$

From (3.4) and (3.5) it follows that in the undeformed state $\bar{x} = \bar{X}$ the pressure p represents a hydrostatic pressure distribution: $p = (\rho g + 2m)X_1$.

4. Non-linear statics

Plane strain

Firstly a basic solution of static plane strain deformation in planes parallel to the X_3-axis is considered.

Cylindrical co-ordinates are introduced according to:

$$x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = z,$$

$$X_1 = R\cos\Theta,\ X_2 = R\sin\Theta,\ X_3 = Z, \tag{4.1}$$

where $0 \leq R \leq \infty$, $-(\pi/2) \leq \Theta \leq (\pi/2)$. See Figure 1.

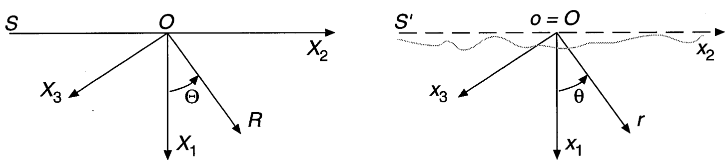

Figure 1. Cartesian co-ordinates and cylindrical co-ordinates.

The deformation is assumed to be

$$r^2 - R^2 = A,\ \theta = \Theta,\ z = Z \tag{4.2}$$

The material deformation gradient tensor $\bar{\bar{F}}$ takes the form (compare (2.5a)):

$$\overline{\overline{\mathrm{F}}} = \left[\frac{\partial x_k}{\partial X_m}\right] = \begin{bmatrix} \frac{\mathrm{d}r}{\mathrm{d}R} & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.3}$$

From (4.2) it follows after differentiation that

$$\frac{\mathrm{d}r}{\mathrm{d}R} \cdot \frac{r}{R} = 1 \tag{4.4}$$

so that $\det(\overline{\overline{\mathrm{F}}}) = 1$ and the deformation is isochoric indeed.
With (3.4) and $C = 0$ the constitutive equations (3.1) take the form

$$\begin{aligned}
\sigma_{rr} &= -(\rho g + 2m) r \cos\theta + 2mR\cos\theta \cdot \frac{R}{r} \\
&= -\rho g r \cos\theta - 2m\cos\theta \cdot \frac{A}{r} \\[1em]
\sigma_{\theta\theta} &= -(\rho g + 2m) r \cos\theta + 2mR\cos\theta \cdot \frac{r}{R} \\
&= -\rho g \cdot r \cos\theta \\[1em]
\sigma_{zz} &= -(\rho g + 2m) r \cos\theta + 2mR\cos\theta \\
\sigma_{r\theta} &= \sigma_{\theta r} = 0, \ \ \sigma_{\theta z} = \sigma_{z\theta} = 0, \ \ \sigma_{rz} = \sigma_{zr} = 0
\end{aligned} \tag{4.5}$$

It should be noted that components of stress satisfy of themselves the conditions of equilibrium, which now take the form:

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho g\cos\theta = 0 \\
&\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} - \rho g\sin\theta = 0 \\
&\frac{\partial \sigma_{zz}}{\partial z} = 0
\end{aligned} \tag{4.6}$$

Rigid semi-cylindrical punch
The first application refers to the case $A > 0$ in (4.2). This application turns out to represent the case of a weightless rigid semi-cylinder of radius $r_0 = A^{1/2}$ which is pressed against the upper surface without friction in such a way that the final upper boundary including the punch is flat (figure 2).
With $A = r_0^2$ the deformation (4.2) takes the form:

$$r^2 - R^2 = r_0^2, \ \ \theta = \Theta, \ \ z = Z \tag{4.7}$$

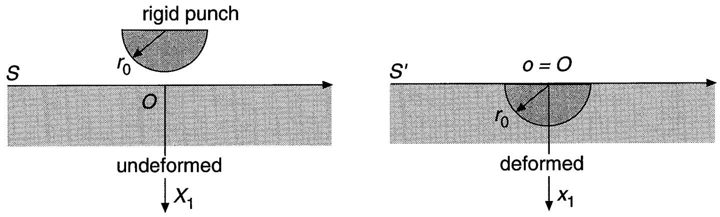

Figure 2. Pressed rigid semi-cylinder.

The condition of equilibrium (3.4) reads with $C = 0$:

$$p = (\rho g + 2m)r\cos\theta \tag{4.8}$$

With (4.5), the components of the stress are given by

$$\begin{aligned}
&\sigma_{rr} = -\rho g r \cos\theta - 2m\frac{r_0^2}{r}\cos\theta \\
&\sigma_{\theta\theta} = -\rho g r \cos\theta \\
&\sigma_{zz} = -(\rho g + 2m)r \cos\theta + 2mR\cos\theta \\
&\sigma_{r\theta} = \sigma_{\theta r} = 0,\ \ \sigma_{z\theta} = \sigma_{\theta z} = 0,\ \ \sigma_{zr} = \sigma_{rz} = 0
\end{aligned} \tag{4.9}$$

so that the boundary conditions of zero stress at $\theta = \pm\pi/2$, $r \geq r_0$ are satisfied. The components of stress (4.9) must balance the pressing downward line force P per unit length of the punch and the body force due to own weight. The horizontal and vertical equilibrium yield, respectively,

$$-2\int_0^{\pi/2} (\sigma_{rr}\sin\theta - \sigma_{r\theta}\cos\theta)r\mathrm{d}\theta = 0 \tag{4.10a}$$

$$-2\int_0^{\pi/2} (\sigma_{rr}\cos\theta + \sigma_{r\theta}\sin\theta)r\mathrm{d}\theta = \frac{\pi}{2}(\rho g + 2m)r_0^2 + \frac{\pi}{2}\rho g(r^2 - r_0^2) \tag{4.10b}$$

The moment equilibrium gives

$$2\int_0^{\pi/2} r\sigma_{r\theta}r\mathrm{d}\theta = 0 \tag{4.10c}$$

From the vertical equilibrium condition (4.10.b) it follows with $r = r_0$ that the downward pressing force P per unit length is given by:

$$P = (\rho g + 2m)\frac{\pi}{2}r_0^2 \tag{4.11}$$

We notice that in the absence of rigidity, i.e. $m = 0$, the line force P just balances the weight of the displaced incompressible perfect fluid. From the expressions (4.8) and (4.9) it follows that the stress distribution at the location of the cylindrical part of the punch $r = r_o$, so that $R = 0$, is given by:

$$\begin{aligned} &p(r = r_0) = (\rho g + 2m) r_0 \cos\theta \\ &\sigma_{rr}(r = r_0) = \sigma_{zz}(r = r_0) = -p(r = r_0) \\ &\sigma_{\theta\theta} = -\rho g r \cos\theta \end{aligned} \tag{4.12}$$

Thus at the punch boundary the state of stress is not isotropic. As a consequence on the location $r = r_0$ there are material planes which must bear shear stress, while the rigidity there is zero. However, the surface $r = r_0$ arises from the line $R = 0$, so that the deformation at $R = 0$ is singular. Due to this singularity material planes at the deformed surface $r = r_0$ with zero rigidity should transmit shear.

Cylindrical notch

The second application refers to the case $A < 0$ in (4.2) and it turns out to represent a semi-cylindrical notch in the deformed configuration (figure 3).
With $(-A) > 0$ we put

$$-A = \frac{\rho g}{2m} r_0^2 \tag{4.13}$$

in which r_0 is the radius of the semi-cylindrical notch in the deformed configuration (figure 3). With (4.13) the deformation (4.2) takes the form

$$R^2 - r^2 = \frac{\rho g}{2m} r_0^2, \ \theta = \Theta, \ z = Z \tag{4.14}$$

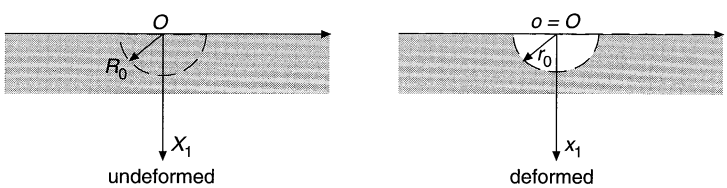

Figure 3. Cylindrical excavation.

The components of stress in the final configuration are (compare (4.5))

$$
\begin{aligned}
&\sigma_{rr} = -\rho g r \cos\theta + \frac{\rho g r_0^2}{r}\cos\theta \\
&\sigma_{\theta\theta} = -\rho g r \cos\theta \\
&\sigma_{zz} = -\rho g r \cos\theta + 2mr\cos\theta\left\{(1 + \frac{\rho g r_0^2}{2mr^2})^{1/2} - 1\right\} \\
&\sigma_{r\theta} = \sigma_{\theta r} = 0,\ \sigma_{\theta z} = \sigma_{z\theta} = 0,\ \sigma_{zr} = \sigma_{rz} = 0
\end{aligned}
\tag{4.15}
$$

It can be verified that the upper boundary of the deformed configuration is free of stresses. Along the semi-cylindrical bottom $\sigma_{rr} = \sigma_{r\theta} = 0$ and along the plane portion of the boundary $\sigma_{\theta\theta} = \sigma_{\theta r} = 0$.

Conformly the transformation (4.14) a semi-circle of radius r_o in the deformed configuration corresponds to a a semi-circle of radius $R_0 = r_0\,(1 + (\rho g/2m))^{1/2}$ in the undeformed configuration. In this way the semi-cylindrical notch with radius r_o can be considered as the result of the removal from the undeformed body of semi-cylinder of material with radius R_0.

On the other hand we may reason as follows (Sparenberg (1996)). According to the transformation (4.14) the removal of a semi-cylinder with radius $R_0{}^* = (\rho g/2m)^{1/2} r_0$ from the undeformed body gives rise to a deformed elastic half-space with a plane upper-boundary, because r becomes imaginary for values of $R < R_0{}^*$. This removal gives rise to the same stress distribution (4.15). A subsequent removal of a semi-cylinder with radius r_0 from the deformed elastic half-space with a plane upper surface leads to the semi-cylindrical notch with radius r_0 in the deformed configuration, since $R_0{}^{*2} + r_0{}^2 = r_0^2\,(2m + \rho g)/2m = R_0^2$. (We notice that the stress distribution in the deformed half-space with a plane upper boundary is singular at the point $r = 0$).

Sparenberg remarks that the way in which the final notch of radius r_0 can be effected, is not unique.

Spherical deformations

It is noticed that the solutions of the corresponding spherical cases of a pressed rigid punch and an excavation may be obtained by considering the family of deformations

$$
r^3 - R^3 = B, \quad \theta = \Theta,\ \varphi = \Phi \tag{4.16}
$$

where (r,θ,ϕ) and (R,Θ,Φ) are the spatial and material spherical co-ordinates, respectively, and B is constant. The deformation gradient tensor $\underline{\underline{\mathrm{F}}}$ has the form

$$
\underline{\underline{\mathrm{F}}} = \begin{bmatrix} \frac{\mathrm{d}r}{\mathrm{d}R} & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ 0 & 0 & \frac{r}{R} \end{bmatrix} \tag{4.17}
$$

and the stress tensor has the form

$$\overline{\overline{\sigma}} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\varphi\varphi} \end{bmatrix} \tag{4.18}$$

The condition of incompressibility, det $(\overline{\overline{\mathrm{F}}}) = 1$, is given by

$$\frac{\mathrm{d}r}{\mathrm{d}R} \cdot \frac{r^2}{R^2} = 1 \tag{4.19}$$

We restrict ourselves to the punch problem so that (4.16) takes the form

$$r^3 - R^3 = r_0^{\ 3} \tag{4.20}$$

The constitutive equations (3.1) become

$$\begin{aligned} \sigma_{rr} &= -p + 2mR\cos\theta \frac{\mathrm{d}r}{\mathrm{d}R} \\ &= -p + 2mr\cos\theta - 2m\cos\theta \frac{r_0^{\ 3}}{r^2} \\ \sigma_{\theta\theta} &= -p + 2mR\cos\theta \frac{r}{R} \\ \sigma_{\varphi\varphi} &= -p + 2mr\cos\theta = \sigma_{\theta\theta} \\ \sigma_{r\theta} &= \sigma_{\theta\varphi} = \sigma_{\theta r} = 0 \end{aligned} \tag{4.21}$$

The equations of equilibrium are (Malvern (1969), p.671),

$$\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} + \rho g\cos\theta = 0 \\ &\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\cos\theta - \rho g\sin\theta = 0 \\ &\frac{1}{r\sin\theta}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} = 0 \end{aligned} \tag{4.22}$$

Elimination of the stresses between (4.21) and (4.22) and integration of the first equation of motion gives

$$p = (\rho g + 2m) r\cos\theta \tag{4.23}$$

and

$$\begin{aligned} \sigma_{rr} &= -\rho g r\cos\theta - 2m\cos\theta \frac{r_0^{\ 3}}{r^2} \\ \sigma_{\theta\theta} &= \sigma_{\varphi\varphi} = -\rho g r\cos\theta \\ \sigma_{r\theta} &= \sigma_{\theta\varphi} = \sigma_{\varphi r} = 0 \end{aligned} \tag{4.24}$$

The components of stress satisfy the boundary conditions of zero stress at $\theta = \pm\pi/2$ and $r \geq r_0$. The pressing punch force P is given by

$$P = (\rho g + 2m)\frac{2\pi}{3} r_0^3 \tag{4.25}$$

Mathematical accidents

From the linear theory of elasticity it is known that due to normal surface loading the stress distribution in the Gibson soil has exactly the same form as the stress distribution in a homogeneous (incompressible) isotropic elastic half-space. This rather surprising result has been established by Lekhnitskii (1962) for the Flamant problem of a vertical line force and the Boussinesq problem of the vertical concentrated force, and therefore holds for any normal loading on basis of the principle of superposition. The awareness of these mathematical accidents may be useful. It can be shown that when the radius of the punch is very small and the quadratic stretch $\lambda^2 - 1$ is replaced by the linear stretch $\lambda - 1$ the non-linear stress distribution reduces to the Flamant or Boussinesq distribution which become now singular at the location of line force and point force application.

It is also purely accidental that the linearisation of the notch problem leads to a stress distribution which is similar to the stress distribution belonging to the removal of a semi-cylinder (-sphere) from a homogeneous (incompressible) isotropic elastic half-space when calculated with the linear theory.

Static subgrade reaction theory

The governing equations and the solved problems show that a downward static displacement w of a point of the half-space is proportional to an additional isotropic stress $-\Delta p$ at that point according to:

$$\Delta p = (\rho g + 2m)w \tag{4.26}$$

From the linear theory it is well-known that the settlement w_0 at some point of the plane upper surface is proportional to the intensity $-q_0$ of the normal stress on the upper surface at that point according to:

$$q_0 = (\rho g + 2m)w_0 \tag{4.27}$$

In this case the stress at the location of the loaded area is *purely isotropic*. From experimental data in soil mechanics and from the linearised deformation theory of a homogeneus isotropic elastic half-space it is known that the stress distribution under a rigid plate that undergoes a uniform settlement, is not uniform so that the theory of subgrade reaction according to (4.27) is not satisfied.

Barkan ((1962), p. 28) notices that the weight of footings influences the rigidity of the soil stratum since an increase of pressure gives rise to an increase of rigidity. With an increase in base contact area between the footing and the soil upper surface, a greater depth of soil is affected by the weight of the foundation, and the influence of deeper soil layers on the footing settlement increases. Barkan supposes that an increase in base contact area and the increase with depth of the initial stress due to own soil weight give rise to the validity of the theory of static subgrade reaction for relatively large foundations as a consequence of the increase of the rigidity with depth (see also Terzaghi (1943), p. 396).

5. Concluding remarks

The Stoneley-Gibson-Varga elastic stratum represents the geometrically non-linear counterpart of the linear Gibson elastic half-space.
There exists the mathematical resemblence between the equations of motion governing the irrotational wave motion through the Stoneley-Gibson-Varga stratum and the equations of motion governing the irrotational deep water motion.
The static downward settlement w of a point of the stratum due to external loading is proportional to an additional pressure $-\Delta p$ at that point according to

$$\Delta p = (\rho g + 2m)w$$

This result is exemplified for the cases of a punch indentation and an excavation. The stress distribution in the stratum is not isotropic due to the shear rigidity of the stratum.

Appendix 1. Historical review

An isotropic, elastic (half-) space is quite appropriate as a seismological model of the earth. Effects of possible departures from this ideal model do not affect the cardinal characteristics of seismological phenomena (Bullen (1967), p. 85).
In seismology two types of surface waves are distinguished, namely, Rayleigh waves and Love waves.
In Rayleigh waves the displacements of the particles consist of two components, one vertical and one horizontal, parallel to the direction of wave propagation. In Love waves the displacements of the particles are parallel to the free surface and pendicular to the direction of wave propagation. Apart from this distinction in wave polarisation, there is a fundamental distinction between Rayleigh waves and Love waves, inasmuch as the former can exist in a homogeneous half-space, whereas the latter can exist only in a non-homogeneous half-space in which the shear modulus, and as a consequence the velocity of the distortional waves, increases with depth. This inhomogeneity of the shear modulus causes the dispersion of the Rayleigh waves, whereas it may involve the very existence or non-extistence of Love waves, that are always dispersive.
E. Meissner (1921) analysed the Love-type waves for a quadratic and linear increase of the shear modulus with increasing depth and for density increasing with depth (see, for example, Vardoulakis (1981) and (1982)).
R. Stoneley (1934) analysed Rayleigh-type waves in an incompressible, isotropic, elastic half-space of a constant density and with a shear modulus μ increasing linearly with depth X_1 according to the law $\mu = \mu_0 + mX_1$.

When we analyse a group of plane harmonic waves of wavelength $2\pi/\kappa$ travelling in the horizontal direction with velocity c, there are the following fundamental parameters (Stoneley (1934), Kruijtzer (1976)):

$$q = \frac{c^2 \cdot \kappa \cdot \rho}{2m}, \quad a = \frac{2 \cdot \kappa \cdot \mu_0}{m}, \quad \frac{4q}{a} = \frac{c^2}{c_s^2}, \quad c_s^2 = \frac{\mu_0}{\rho} \tag{a.1.1}$$

where ρ is the constant mass density of the Stoneley half-space and c_s is the velocity of propagation of the distortional wave (S-wave) being transmitted through an unbounded homogeneous medium. When we assume that in the Stoneley half-space acts an original hydrostatic stress distribution due to a constant gravity, that increases linearly with depth, then the dispersion relation of the waves is given by (Kruijtzer (1976)):

$$-m^2 \cdot a^2 \cdot U(1-q, 1, a) + m(\rho g + 2m - 2mq + 2ma) \cdot U(-q,0,a) = 0 \tag{a.1.2}$$

where g is the acceleration due to gravity and U are Kummer functions. In the absence of the superimposed original hydrostatic stress distribution the relation may be reduced to Stoneley's dispersion relation in terms of Whittaker functions. The Kummer functions and Whittaker functions, both confluent hypergeometric functions, are closely related.
It is historically noteworthy that at the time of Stoneley's investigations Olver's theorems on the asymptotic expansion of the Kummer and Whittaker functions were not yet available (see Slater (1960)). These theorems guarantee a complete investigation of the dispersion relation (a.1.2) (Braaksma, 1975).
If in (a.1.2) both q and a, with $1 \leq (a/4q) < \infty$, i.e., $1 \leq (c_s^2/c^2) < \infty$, are allowed to approach infinity, the relation (a.1.2) reduces to the well-known Biot relation (Biot (1964)):

$$-\frac{\rho \cdot g \cdot c^2}{\kappa \cdot \mu_0 \cdot c_s^2} + (2 - \frac{c^2}{c_s^2})^2 = 4(1 - \frac{c^2}{c_s^2})^{1/2} \tag{a.1.3}$$

This relation (a.13) gives the velocity of wave propagation of Raylaigh waves transmitted over the upper surface of an incompressible homogeneous, isotropic elastic half-space being subjected to an initial hydrostatic stress distribution due to a constant gravity: $0.955 \leq (c/c_s) \leq 1$ when $0 \leq (\rho g/\kappa\mu_0) \leq 1$.

The case of vanishing top rigidity, i.e. $a = 0$, gives rise to various types of waves. In the particular cases of q is an integer the Kummer functions reduce to Laguerre polynomials and each integer q generates its own wave. This result has been re-established partly by Vardoulakis (1981), who gives no account to initial hydrostatic stress distribution to gravity. Meissner (1921), who analysed the Love waves, had already noticed that such a discrete spectrum of waves exists.
In the more general case in which q is not an integer and the top rigidity vanishes, $a = 0$, the relation (a.1.2) reduces to

$$c = \left\{\frac{\rho \cdot g + 2m}{\rho \cdot \kappa}\right\}^{1/2} \qquad \text{(a.1.4)}$$

In this case the wave motion is irrotational and the waves are mathematically similar to the gravitational waves in deep water (as a non-viscous incompressible fluid). Boundary value problems involving a normal stress on the upper surface, give rise to an irrotational motion (Kruijtzer (1976)), since there is a continuous spectrum of waves.
In the case of a vanishing top rigidity the Stoneley half-space reduces to the Gibson half-space. In his famous 1967-paper R.E. Gibson showed that the upper-surface of this half-space, being named after him, reacts under static normal loading like a uniform bed of springs (a Winkler foundation). The effects of non-vanishing top rigidity and of compressibility on this static result have been widely investigated by Awojobi and Gibson (1973) and Brown and Gibson (1972). The effect of finite depth has been investigated by Gibson, Brown and Andrews (1971).
Awojobi (1974, 1982) investigated some elastodynamic boundary problems for the Gibson half-space. He presented his solutions in terms of numerically solved integrals. Some explicit solutions have been presented by Kruijtzer (1976). Rather recent treatises on the statics and the dynamics of a (linearly) non-homogeneous half-space are presented by Vettros (1998) and Muravskii (1997).
The awareness of the linearised irrotational motion gave rise to the search for the corresponding non-linear, irrotational motion.

Appendix 2. Constitutive equations

Three fundamental postulates are assumed to be valid for any constitutive theory of purely mechanical (isothermal or isentropic) phenomena in a continuous medium. The three postulates are (Malvern (1968), p. 379):
1. The Principle of determination for stress: the stress in a body is determined by the history of the motion of the body; in elasticity this history dependence consists only in possessing a natural state to which a body will return upon unloading.
2. The Principle of local action: in determining the stress as a given particle, the motion outside an arbitrary neighborhood of this particle may be disregarded, so that action-at-a-distance stress-strain relations are excluded.
3. Principle of material frame-indifference: constitutive equations must be invariant under changes of frame reference. That is, two observers even if in relative motion with respect to each other, observe the same stress in a given body. This principle holds if the stress-strain relation is not affected by mass effects (Coriolis forces). Truesdell stipulates that Newton's Third Law of action and reaction represents this principle in newtonian terms.

We firstly pay attention to the third principle. We assume that relative motion of the observers consists of rotations, reflections and point reflections because a translation is of minor interest. Such motions are represented by an orthogonal tensor $\bar{\bar{Q}}$ with $\bar{\bar{Q}}^{-1} = \bar{\bar{Q}}^{T}$, *which is a function of the time t*. This change of frame induces the following transformations.
A vector $\bar{v}$ as seen by an observer will be the vector $\bar{v}' = \bar{\bar{Q}} \cdot \bar{v}$ as seen by another observer.
A material line element $d\bar{X}$ in the undistorted body becomes the line element $d\bar{x}$ in the deformed

body accordingly $d\bar{x} = \bar{\bar{F}} \cdot d\bar{X}$. In the same way $d\bar{x}' = \bar{\bar{F}}' \cdot d\bar{X}'$. With $d\bar{X} = d\bar{X}'$ and $d\bar{x}' = \bar{\bar{Q}} \cdot d\bar{x}$, it follows that $\bar{\bar{F}}' = \bar{\bar{Q}} \cdot \bar{\bar{F}}$. Further, let $d\bar{p}$ be the traction vector on the area da with outer normal $\bar{n}$ in the deformed body, so that $d\bar{p} = \bar{\bar{\sigma}} \cdot \bar{n}\, da$. In the same way $d\bar{p}' = \bar{\bar{\sigma}}' \cdot \bar{n}' da$, since scalars are invariant under a change of frame. With $d\bar{p}' = \bar{\bar{Q}} \cdot d\bar{p}$ and $d\bar{n}' = \bar{\bar{Q}} \cdot d\bar{n}$ it follows that $\bar{\bar{\sigma}}' = \bar{\bar{Q}} \cdot \bar{\bar{\sigma}} \cdot \bar{\bar{Q}}^T$.
On the base of the first two principles, the most general constitutive equation of an elastic material is given by

$$\bar{\bar{\sigma}} = \bar{\bar{g}}(\bar{\bar{F}}), \tag{a.2.1}$$

where $\bar{\bar{\sigma}}$ is the true stress in the deformed body and the response function $\bar{\bar{g}}$ is a tensor-valued function of the deformation gradient tensor $\bar{\bar{F}}$.
To ensure frame indifference, the response function $\bar{\bar{g}}(\bar{\bar{F}})$ must satisfy

$$\bar{\bar{g}}(\bar{\bar{Q}} \cdot \bar{\bar{F}}) = \bar{\bar{Q}} \cdot \bar{\bar{g}}\,(\bar{\bar{F}}) \cdot \bar{\bar{Q}}^T, \tag{a.2.2}$$

since $\bar{\bar{F}}$ becomes $\bar{\bar{Q}} \cdot \bar{\bar{F}}$ and $\bar{\bar{\sigma}}$ becomes $\bar{\bar{Q}} \cdot \bar{\bar{\sigma}} \cdot \bar{\bar{Q}}^T$. Now let $\bar{\bar{Q}}^T = \bar{\bar{R}}$ represents a local rigid rotation, and $\bar{\bar{F}} = \bar{\bar{R}} \cdot \bar{\bar{U}}$ with $\bar{\bar{U}} = \bar{\bar{U}}^T$, then the relation (a.2.2) becomes:

$$\bar{\bar{g}}(\bar{\bar{R}}^T \cdot \bar{\bar{F}}) = \bar{\bar{R}}^T \cdot \bar{\bar{g}}(\bar{\bar{F}}) \cdot \bar{\bar{R}},$$

so that with $\bar{\bar{R}}^T \cdot \bar{\bar{F}} = \bar{\bar{R}}^{-1} \cdot \bar{\bar{F}} = \bar{\bar{U}}$:

$$\bar{\bar{\sigma}} = \bar{\bar{g}}(\bar{\bar{F}}) = \bar{\bar{R}} \cdot \bar{\bar{g}}(\bar{\bar{U}}) \cdot \bar{\bar{R}}^T \tag{a.2.3}$$

Thus frame-indifference requires that in equation (a.2.3) the dependence on $\bar{\bar{F}}$ must take the form of an arbitrary function of the right stretch tensor $\bar{\bar{U}}$ with an additional explicit dependence on the local rigid rotation $\bar{\bar{R}}$. It may be noticed that the symmetric Piola-Kirchhoff stress $\bar{\bar{\sigma}}' = (\det \bar{\bar{F}})\, \bar{\bar{F}}^{-1} \cdot \bar{\bar{\sigma}} \cdot (\bar{\bar{F}}^{-1})^T$ possesses the independence on $\bar{\bar{R}}$, i.e. $\bar{\bar{\sigma}} = \bar{\bar{h}}(\bar{\bar{U}})$, independently of either isotropy or non-isotropy of the elastic material.

The isotropy or material symmetry group is the group of all *static* density-preserving either actual or non-actual deformations of the material reference configuration into a new material reference configuration such the material at a particle *P* in the original configuration and the material at the same particle *P* in the new configuration are indistinguishable from each other in their response to the same loading. The definition of an isotropic simple material, e.g., an isotropic elastic material, is given in terms of the full orthogonal group with members $\bar{\bar{Q}}$ with $\bar{\bar{Q}}^{-1} = \bar{\bar{Q}}^T$ and $\det(\bar{\bar{Q}}) = \pm 1$, including $\bar{\bar{1}}$ and $-\bar{\bar{1}}$. As a consequence, for isotropic materials

$$\bar{\bar{g}}(\bar{\bar{F}}) = \bar{\bar{g}}(\bar{\bar{F}} \cdot \bar{\bar{Q}}) \quad \text{or} \quad \bar{\bar{g}}(\bar{\bar{F}}) = \bar{\bar{g}}(\bar{\bar{F}} \cdot \bar{\bar{Q}}^T). \tag{a.2.4}$$

Since (a.2.4) holds for any non-singular $\bar{\bar{F}}$ including $\bar{\bar{Q}} \cdot \bar{\bar{F}}$ for any $\bar{\bar{F}}$, we may write (a.2.4) in the form:

$$\bar{\bar{g}}(\bar{\bar{Q}} \cdot \bar{\bar{F}}) = \bar{\bar{g}}(\bar{\bar{Q}} \cdot \bar{\bar{F}} \cdot \bar{\bar{Q}}^T). \tag{a.2.5}$$

With use of (a.2.2), with $\bar{\bar{Q}}$ fixed in time, we see that an elastic body is isotropic if and only if its response function $\bar{\bar{g}}(\bar{\bar{F}})$ relative to some undistorted state satisfies the identity:

$$\bar{\bar{Q}} \cdot \bar{\bar{g}}(\bar{\bar{F}}) \cdot \bar{\bar{Q}}^T = \bar{\bar{g}}(\bar{\bar{Q}} \cdot \bar{\bar{F}} \cdot \bar{\bar{Q}}^T). \tag{a.2.6}$$

With, for example, $\bar{\bar{Q}} = \bar{\bar{R}}$ and $\bar{\bar{F}} = \bar{\bar{U}}$ the relation (a.2.6) becomes

$$\bar{\bar{R}} \cdot \bar{\bar{g}}(\bar{\bar{U}}) \cdot \bar{\bar{R}}^T = \bar{\bar{g}}(\bar{\bar{R}} \cdot \bar{\bar{U}} \cdot \bar{\bar{R}}^T) = \bar{\bar{g}}(\bar{\bar{V}}), \quad \bar{\bar{F}} = \bar{\bar{R}} \cdot \bar{\bar{U}} = \bar{\bar{V}} \cdot \bar{\bar{R}}. \tag{a.2.7}$$

Combination of the relations (a.2.3) and (a.2.7) gives

$$\bar{\bar{\sigma}} = \bar{\bar{g}}(\bar{\bar{V}}) \tag{a.2.8}$$

This dependency of the actual stress in the deformed body on the left stretch tensor $\bar{\bar{V}}$ holds only for isotropic simple materials (Malvern (1970), p. 391).
We now derive the expression for the strain-energy function W of an incompressible Varga material. We follow the method of Rivlin (1948).
The strain ellipsoid has the property that the length of its radius in any direction is proportional to the stretch for an element that lies in that direction in the deformed state. Let us suppose that a cube of unit edge in the unstrained state is strained in such a manner that in the deformed state it is a cube whose edges are parallel to the axes of the strain ellipsoid and have lengths λ_1, λ_2 and λ_3, respectively, such that $\lambda_1\lambda_2\lambda_3 = 1$, satisfying the incompressibility condition. Further, let us suppose that the deformation is a pure, homogeneous strain under action of three pairs of equal and oppositely-directed forces, f_1, f_2 and f_3, mutually at right angles. With (compare the constitutive equations (2.10)):

$$\sigma_{11} = -p + 2\mu\lambda_1\,,\ f_1 = -p\lambda_2\lambda_3 + 2\mu = \frac{-p}{\lambda_1} + 2\mu,$$

$$\sigma_{22} = -p + 2\mu\lambda_2\,,\ f_2 = -p\lambda_1\lambda_3 + 2\mu = \frac{-p}{\lambda_2} + 2\mu,$$

$$\sigma_{33} = -p + 2\mu\lambda_3\,,\ f_3 = -p\lambda_1\lambda_2 + 2\mu = \frac{-p}{\lambda_3} + 2\mu.$$

The work W done, in straining the material quasi-statically from dimensions $1 \times 1 \times 1$ to $\lambda_1 \times \lambda_2 \times \lambda_3$ is given by

$$W = p\left\{\int_1^{\lambda_1} \frac{-1}{\lambda_1}\mathrm{d}\lambda_1 + \int_1^{\lambda_2} \frac{-1}{\lambda_2}\mathrm{d}\lambda_2 + \int_1^{\lambda_3} \frac{-1}{\lambda_3}\mathrm{d}\lambda_3\right\} + 2\mu\left\{\int_1^{\lambda_1}\mathrm{d}\lambda_1 + \int_1^{\lambda_2}\mathrm{d}\lambda_2 + \int_1^{\lambda_3}\mathrm{d}\lambda_3\right\}$$

Thus,

$$W = 2\mu\,(\lambda_1 + \lambda_2 + \lambda_3 - 3).$$

since $\lambda_1\lambda_2\lambda_3 = 1$. We notice that $W \geq 0$, since the arithmic mean $(\lambda_1 + \lambda_2 + \lambda_3)/3$ is always larger than the geometric mean $(\lambda_1\lambda_2\lambda_3)^{1/3} = 1$, as mathematics learns.
The work W is the stored energy per unit volume of the strained material with the principal extensions are $\lambda_1 - 1$, $\lambda_2 - 1$, $\lambda_3 - 1$. Finally we notice that $(\lambda_1 + \lambda_2 + \lambda_3)$ is the invariant trace of $\overline{\overline{\mathrm{V}}}$.
Rivlin's incompressible, isotropic neo-Hookean material possesses the strain-energy function

$$W = \mu\,(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2 - 3),$$

where $\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2$ is the trace of $\overline{\overline{\mathrm{B}}} = \overline{\overline{\mathrm{F}}} \cdot \overline{\overline{\mathrm{F}}}^{\mathrm{T}} = \overline{\overline{\mathrm{V}}}^2$.

Appendix 3.

In the equation of motion (2.8) into the x_1-direction we consider the term

$$A = \frac{\partial\sigma_{x_1x_1}}{\partial x_1} + \frac{\partial\sigma_{x_2x_1}}{\partial x_2} + \frac{\partial\sigma_{x_3x_1}}{\partial x_3} \qquad \text{(a.3.1)}$$

with according (3.1):

$$\sigma_{x_1x_1} = -p + 2mX_1\frac{\partial x_1}{\partial X_1},\ \sigma_{x_2x_1} = 2mX_1\frac{\partial x_2}{\partial X_1},\ \sigma_{x_3x_1} = 2mX_1\frac{\partial x_3}{\partial X_1} \qquad \text{(a.3.2)}$$

Elimination of the stresses between (a.3.1) and (a.3.2) gives

$$\begin{aligned} A = -\frac{\partial p}{\partial x_1} + 2m&\left\{\frac{\partial X_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial X_1} + \frac{\partial X_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial X_1} + \frac{\partial X_1}{\partial x_3}\cdot\frac{\partial x_3}{\partial X_1}\right\} \\ 2mX_1&\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial x_1}{\partial X_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial x_2}{\partial X_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial x_3}{\partial X_1}\right)\right\} \end{aligned} \qquad \text{(a.3.3)}$$

From (2.5c) it follows that

$$\frac{\partial X_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial X_1} + \frac{\partial X_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial X_1} + \frac{\partial X_1}{\partial x_3}\cdot\frac{\partial x_3}{\partial X_1} = 1 \qquad \text{(a.3.4)}$$

Further, since $\overline{\overline{\mathrm{F}}} \cdot \overline{\overline{\mathrm{F}}}^{-1} = \overline{\overline{1}}$ and $\det(\overline{\overline{\mathrm{F}}}) = 1$:

$$\frac{\partial x_1}{\partial X_1} = \frac{\partial X_2}{\partial x_2} \cdot \frac{\partial X_3}{\partial x_3} - \frac{\partial X_3}{\partial x_2} \cdot \frac{\partial X_2}{\partial x_3} \qquad \text{(a.3.5a)}$$

where the right hand side is the co-factor of $\partial X_1 / \partial x_1$. Similarly:

$$\frac{\partial x_2}{\partial X_1} = -\left\{ \frac{\partial X_1}{\partial x_2} \cdot \frac{\partial X_3}{\partial x_3} - \frac{\partial X_1}{\partial x_3} \cdot \frac{\partial X_3}{\partial x_2} \right\} \qquad \text{(a.3.5b)}$$

$$\frac{\partial x_3}{\partial X_1} = \frac{\partial X_1}{\partial x_2} \cdot \frac{\partial X_2}{\partial x_3} - \frac{\partial X_2}{\partial x_2} \cdot \frac{\partial X_1}{\partial x_3} \qquad \text{(a.3.5c)}$$

Substitution of (a.3.4) and (a.3.5) into the expression (a.3.3) gives

$$\begin{aligned} A = -\frac{\partial p}{\partial x_1} + 2m + 2mX_1 \Bigg\{ & \frac{\partial}{\partial x_1}\left(\frac{\partial X_2}{\partial x_2} \cdot \frac{\partial X_3}{\partial x_3} - \frac{\partial X_3}{\partial x_2} \cdot \frac{\partial X_2}{\partial x_3} \right) \\ & - \frac{\partial}{\partial x_2}\left(\frac{\partial X_1}{\partial x_2} \cdot \frac{\partial X_3}{\partial x_3} - \frac{\partial X_1}{\partial x_3} \cdot \frac{\partial X_3}{\partial x_2} \right) \\ & + \frac{\partial}{\partial x_3}\left(\frac{\partial X_1}{\partial x_2} \cdot \frac{\partial X_2}{\partial x_3} - \frac{\partial X_2}{\partial x_2} \cdot \frac{\partial X_1}{\partial x_3} \right) \Bigg\} \end{aligned} \qquad \text{(a.3.6)}$$

Since $\overline{\overline{\mathrm{F}}}$ is symmetric: $\partial X_i / \partial x_k = \partial X_k / \partial x_i$, the third term of (a.3.6) vanishes, so that the expression (a.3.1), being equal to the expressions (a.3.3) and (a.3.6), reduces

$$A = -\frac{\partial p}{\partial x_1} + 2m \qquad \text{(a.3.7)}$$

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Litterature

(1921) Meissner, E.,
Elastische Oberflächenwellen mit Dispersion in einem inhomogenen Medium,
Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich, 66, 181-195, (1921).

(1934) Stoneley, R.,
The transmission of Rayleigh waves in a heterogeneous medium, Geophysical Supplements to the Monthly Notices of the Royal Astronomical Society, London, 3, 222-232, (1934).

(1943) Terzaghi, K.,

Theoretical Soil Mechanics, John Wiley & Sons, Inc, NY (1943, 1966).

(1945) Lamb, H.,
Hydrodynamics, Dover Publications Inc., N.Y., (1945).

(1948) Rivlin, R.S.,
Large elastic deformations of isotropic materials, Phil. Trans. R. Soc., London, Vol. 240, A.822, p. 459-526, (1948).

(1960) Slater, L.J.,
Confluent hypergeometric functions, Cambridge Univ. Press, Cambridge, (1960).

(1962) Barkan, D.D.,
Dynamics of Bases and Foundations, McGraw-Hill Book Company, (1962).

(1962) Lekhnitski, S.G.,
Prikl.Mat.Mekh., 26, p. 1446-1453, (1962).

(1964) Biot, M.A.,
Mechanics of Incremental Deformations, John Wiley and Sons Inc., N.Y., (1964).

(1965) Truesdell, C. and W. Noll,
Non-linear Field Theories of Mechanics, Handbuch der Fysik, ed. S. Flügge, Berlin, Springer-Verlag, (1965). Vol. III 3, sect 61.

(1965) Bullen, K.E.
An Introduction to the Theory of Seismology, Cambridge Univ. Press, Cambridge (1965).

(1966) Varga, O.H.,
Stress-strain behaviour of elastic materials, Interscience, N.Y., (1966).

(1967) Gibson, R.E.,
Some Results Concerning Displacements and Stresses in a Non-Homogeneous Elastic Half-Space, Geotechnique 17, 58-67; 18, 275; 19, 160, (1967)

(1969) Malvern, L.E.,
Introduction to the Mechanics of a continuous medium, Prentice Hall Inc., Englewood Cliffs, New Yersey, (1969).

(1970) Richart, F.E., R.D. Woods and J.R. Hall,
Vibrations of Soils and Foundations, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1970).

(1971) Gibson, R.E., P.T. Brown and K.R.F. Andrews,
Some Results Concerning Displacements and Stresses in a Non-Homogeneous Elastic Layer, ZAMP, Vol. 22, Fasc. 5, p. 855-864, (1971).

(1972) Brown, P.T. and R.E. Gibson,
Surface Settlements of a Deep Elastic Stratum whose Modulus Increases Linearly with Depth, Canadian Geotechical Journal, Vol. 9, no. 4, p. 467-476, (1972).

(1973) Awojobi, A.O.,
Vibration of rigid bodies on non-homogeneous semi-infinite elastic media, Q. Jl. Mech. Appl. Math., Vol. XXVI, Pt. 4, 1973.

(1973) Awojobi, A.O. and R.E. Gibson,
Plane Strain and Axially Symmetric Problems of A Linearly Non-Homogeneous Elastic Half-space, Quarterly Journal of Mechanics and Applied Mathematics, Vol. XXVI, Part 3, p. 285-302, (1973).

(1974) Awojobi, A.O.
Vertical vibration of a rigid circular foundation on Gibson soil, Geotechnique 22, p. 333-343, (1974).
Awojobi, A.O.
Harmonic rocking of rectangular foundations on generalised Gibson stratum, Geotechnique 24, p. 655-659, (1974).

(1975) Braaksma, B.,
University of Groningen, The Netherlands, privte communication (1975)

(1976) Kruijtzer, G.F.J.,
Dynamics of a linearly inhomogeneous incompressible isotropic elstic half-space, Applied Scientific Research, 32, p. 1-29, (1976).

(1981) Vardoulakis, I.,
Suface waves in a Half-space of Submerged Sand, Earthquake Engineering and Structural Dynamics, Vol. 9, p. 329-342, (1981).

(1982) Awojobi, A.O.,
Vertical vibration of a compound half-space of homogeneous stratum with Gibson subsoil, Geotechnique, 32, no. 4, 305-313 (1982) (See also Stoneley (1934))

(1982) Vardoulakis, I.,
Torsional Surface Waves in Inhomogeneous Elastic Media, Depart. of Civil Engin., Minneapolis, MN 55455, U.S.A., (1982).

(1982) Spencer, A.J.M.,
The University of Nottingham, U.K., private communication (1982).

(1985) Gazetas, G. and E.T. Selig.
Vibration Problems in Geotechnical Engineering, Proceeding of a Symposium, American Society of Civil Engineers, 1985.

(1985) Wolf, J.P.,
Dynamic soil-structure interaction, Englewood Cliffs: Prentice Hall, 1985.

(1996) Sparenberg, J.A.,
University of Groningen, The Netherlands, private communication (1996).

(1997) Muravskii, G.
1. *Green function for an incompressible linearly non-homogeneous half-space*, Archive of Appl. Mech. 67 (1996), 81-95.
2. *Green functions for a compressible linearly non-homogeneous half-space*, Archive of Appl. Mech. 67 (1997), 521-534.

(1998) Vettros, C.,
Elastic settlement and rotation of rectangular footings on non-homogeneous soils, Geotechnique 48, no. 5, p. 703-707, 1998.