Standard linear solid model for dynamic and time dependent behaviour of building materials

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Vibrations in building structures are almost always undesirable. Whether in the form of low frequency oscillations, or droning of the structure, or as audible noise, they may effect the comfort of the user. They may even effect the safety and the integrity of (parts of) the structure. Damping of mechanical vibrations has always been a point of consideration for the engineer. Presently, it is of increasing interest and importance for the building industry. There are several reasons for this: (1) increasing use of high quality materials which exhibit very low material damping; (2) tendency towards "slender" designing requiring a lower volume of (high quality) materials, thus offering less capacity for absorption of vibration energy; (3) excessive noise and vibration hindrance and interference in highly populated area's (particularly city centers), due to high concentration of structures including high rise buildings, surface- or underground infrastructure, combined with replacement of natural soil between them by low damping solid material; (4) necessity to meet higher performance standards and environmental requirements as to vibration hindrance and noise; (5) necessity to apply high vibration-absorbing materials and systems. Study and careful description of various sources and forms of damping, including the internal damping of materials, is a requirement for meeting these developments.
This paper discusses the merits of a three parameter rheological model for modeling time dependent behaviour of materials in general and dynamic behaviour in particular, with emphasis on damping of free vibrations. This model is known in litterature as the standard linear solid (SLS).
Damping of vibrations in materials is normally anticipated on the basis of the damping modes of the standard mechanical model for damping: viscous damping element parallel with a elastic element (spring). It is shown that these modes are not the general free damping modes for linear materials. When the three-parameter (SLS) model is adopted, the general damping modes do appear. The agreement between model and actual time-dependent material behaviour is acceptable for a wide range of materials with low as well as high damping under both free and forced vibration, over a wide frequency (or time) domain. Parameters can be adapted according to specific materials, including composite materials.
The standard (spring-damper) model retains its applicability for low-damping materials.

Key words: **Internal damping, solid damping, free damped vibrations, dynamic modulus, loss factor, relaxation, standard linear solids, composite materials**

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1. Introduction

Time-dependent and dynamic behaviour of building materials is a subject of on-going interest to the building engineer and so are practical models, parameters, etc. to interpret, characterize, predict and inter-relate the various time dependent phenomena such as creep, stress relaxation and damping.

Mechanical damping of materials is almost invariably discussed in analogy with the damping of a simple spring-mass system with a damping element (dashpot) parallel to the spring. It follows that the well-known hierarchy of free damping modes: periodic damping, critical damping and exponential (non-periodic) damping of this system, is likewise anticipated for free damped vibrations in materials (see for instance ref. 2).

Figure 1.1 Kelvin-model (left) and Maxwell-model, both with a mass attached

This situation is surprising because it has been well-known (ref.1, p.13) for a long time that the two-element spring–dashpot system (elements parallel – the mechanical equivalent of the rheological model according to KELVIN (or VOIGT)) – does not offer a realistic presentation of time-dependent mechanical properties of materials, such as for instance creep and stress relaxation. Only certain thermoplastics with a low degree of cross-linking deform reasonably according to this Kelvin model. Under harmonic loading the KELVIN model predicts a dynamic modulus and a phase shift which will be highly dependent on frequency (§4), whereas the modulus and loss angle of most actual materials show little variation with – or a very smooth dependency of the frequency (ref.1, ref. 2, p. 24, ref. 3, p. 25).

In this paper we will present the damped vibration behaviour of materials following a three-element schematization: the standard linear solid (SLS)- or ZENER-model (Figure 1.2), sometimes termed the "triparameter anelastic model" or "standard anelastic model" (ref.1, p. 13 and 76). This rheological model is known (ref.1, p.13) to offer a realistic representation of actual materials behaviour over the whole frequency range from creep and stress relaxation to dynamic modulus, dynamic loss factor, rate effects and impact loading.

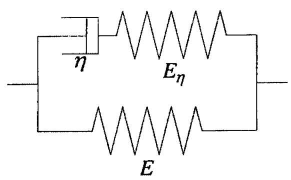

Figure 1.2 Zener-rheological model, or Standard Linear Solid (SLS) model

The value of the (three) parameters of this model can be so adapted as to obtain best "fit" with actual linear non-elastic materials.

2. Modelling constitutive and dynamic behaviour. Survey

Modelling dynamic behaviour of a material or structure requires linking accellerations caused by the action of internal and external forces on its mass, with the time-dependent strains etc. given by a sufficiently precise description of its constitutive behaviour.
A rigorous description (ref. 4) – although not very practical of the time dependent stress-strain relation of linear materials is contained in the homogeneous differental equation (DE):

$$a_0\sigma + a_1\frac{d\sigma}{dt} + a_2\frac{d^2\sigma}{dt^2} + \cdots = b_0\varepsilon + b_1\frac{d\varepsilon}{dt} + b_2\frac{d^2\varepsilon}{dt^2} + \cdots \tag{2.1}$$

in which σ and ε are the stress and the strain tensor respectively.
It can be shown that for simple loading situations: uniaxial, shear, the stress-strain relationship of a so-called Maxwell chain (Figure 2.1 (ref. 4 and 5)) is equivalent with the DE given above. In fact, well-known rheological models such as Hooke, Newton, Maxwell, Kelvin, SLS, Burgers, are simple special cases of both the DE and the Maxwell chain. The branches of the chain are usually arranged according to decreasing relaxation time. Infinite relaxation time ($c_0 = \infty$) of the zero-branch ensures that the material is essentially a solid.

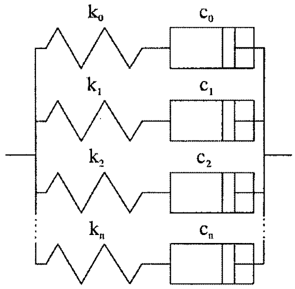

Figure 2.1 Maxwell chain

At the highest strain rates or frequencies all the dampers will exhibit infinite stiffness and the total stiffness of the system is simply $\sum_{i=0}^{n} k_i$, the so-called *unrelaxed stiffness*. At lower rates the dashpots in the branches with lowest relaxation times will relax during loading, and so will the forces in these branches, reaching zero if time permits. At strain rate zero the only stiffness remaining is k_o , *the relaxed stiffness*. It follows that under harmonic loading there will be phase shift between stress and strain which will vanish at both very low and very high frequencies.
Time dependent behaviour of linear visco-elastic material can be approached in principle up to any desired precision by adding sufficient branches, i.e. by incorporating a sufficient number of terms

(Figure 2.1). All this under the assumption that a sufficient number of coefficients (= the values of the branch elements of a sufficient number of branches) are known.

Proposing a scheme like the Maxwell chain is not merely a matter of introducing more constants to obtain better curve fitting. The branches may well correspond with actual relaxation processes in solids. Lazan (ref. 1, p. 39) and Zener (ref. 6) have distinguished a whole range of relaxation mechanisms in metals. It is not the subject of the present publication but we will nevertheless mention a number of such mechanisms:

A. (on the micro-level): *interchange of pairs of solute atoms;* relaxation time 10^{+2} s, independent of specimen size but strongly dependent of temperature;
B. (on the meso-level): *grain boundary displacements;* relaxation time 10^{+8} s, independent of the specimen size;
C. (on the macro-level): *transverse thermal currents* in a specimen subjected to bending, due to the rise and fall of temperature on the tensile and compressive side of the specimen respectively (adiabatic loading) which occurs when the oscillation frequency is sufficiently high; relaxation time of the order of 10^{-2} s, but highly dependent on the dimensions of the specimen or structural member;
D. (on the macro-level): *time dependent displacements* in composites in which elastic constituents (fibres, particles) are embedded in a viscous or visco-elastic matrix;
Appendix IV-2 offers an example with a relaxation time of the order of 3 s at room temperature.

In amorphous solids there is a continuous spectrum of relaxation times (ref. 4).

There are some organic materials for which values of the elements of four or five branches of the Maxwell chain representation of a material have been published, but this is not the case for materials in general. Moreover it would seem that the process of solving the DE(2.1) is far too tedious to be adopted in general engineering practice as a tool for determining time dependent stresses, strains, and displacements of the structure under consideration.
The Maxwell chain idea is nevertheless very useful for understanding time dependent behaviour under different loading conditions. It is clear for instance that mechanism B could show up in long term creep testing, but never in an oscillation experiment; for mechanism C the situation is exactly the other way around. A could show up in short term creep testing but it would require a testing frequency of 10^{-2} c/s to show the effect on the damping of oscillations.

For general engineering practice a constitutive and dynamic description should be favoured which is simple but nevertheless sufficiently sophisticated to bring about the essential features of time dependent behaviour.

Survey of the article
Responses of the standard spring (k)–dashpot (c) system and the SLS system, both with a mass (m) attached, under forced vibrations have been treated extensively in literature (ref. 1) and can also be found in textbooks (with ref. 7 as a very good example); they are summarized for convenience in §4.

§5 addresses the free damped vibrations of both systems. Results for the SLS system are treated in more detail and presented in a manner that has not been published before.
Equations and solutions for both systems are given in §3.
In the discussion (§6) we will compare the outcome for both systems and conclude that the SLS-description would seem to be an attractive compromise between too complex descriptions and models that are too simple to cover damping phenomena in general.
Appendices I and II include a full treatment of the mathematics involved, Appendix III interrelates various time dependent phenomena of SLS-type materials and Appendix IV illustrates the SLS-description for a number of construction materials.

3. Differential equation and parameters

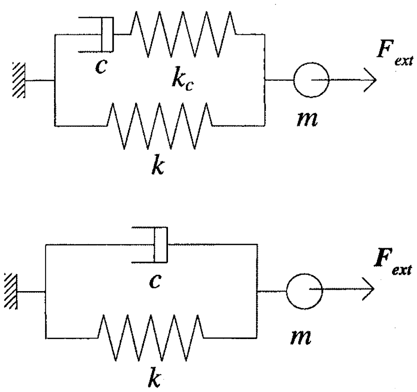

Figure 3.1. Standard Linear Solid (SLS) mechanical model and Kelvin model with a mass attached (kcm-model)

Dynamic behaviour of a construction is obtained by combining:
- the constitutive laws of the constituting materials
- the dynamic response to internal and external forces governed by Newton's law

into *wave* equations which must next be solved for the specific geometry of the structure, its mass distribution and the boundary conditions. When comparing and proposing model descriptions we will limit ourselves to one-dimensional cases with one concentrated mass. In that case the solutions will be vibrations, not waves. Equations will be given in this paragraph. Once a model has proved useful, the extension to structures with distributed mass, although mathematically complex, will be straightforward in principle.

Fig 3.1 shows both models under consideration. For brevity we will term the first model *slsm* and the second *kcm* in this paragraph.

It is worthwhile pointing out that, although the interpretation of simple systems of masses, springs and dashpots is normally one-dimensional (all forces and displacements along the same axis), the correspondence with actual vibrating materials is by no means restricted to uniaxial stresses and strains. So, k, k_c and c may correspond with E, E_c and η[1)], but likewise with G, G_c and the shear viscosity, or with appropriate bending stiffnesses.

In this paragraph we will first study the differential equation of the mechanical system corresponding with a SLS, with a mass attached (*slsm*). The 'mechanical' parameters of this system: k, k_c and c correspond with the moduli E, E_η and the (extensional) viscosity of the material. The correspondence between the mechanical model and a vibrating (SLS-type) material is given and illustrated in Appendices III and IV.

If we combine the constitutive equation (in mechanical formulation) of a SLS-material:

$$F_{\text{int}} + \frac{c}{k_c}\frac{dF_{\text{int}}}{dt} = k\left(u + c\left(\frac{1}{k} + \frac{1}{k_c}\right)\frac{du}{dt}\right) \tag{3.1}$$

with the dynamic equation:

$$m\frac{d^2u}{dt^2} = F_{ext} - F_{int} \tag{3.2}$$

(Figure 3.1) the following third order differential equation of the SLS-type rheological system (*slsm*) can be obtained:

$$\frac{mc}{k_c}\frac{d^3u}{dt^3} + m\frac{d^2u}{dt^2} + c\left(1 + \frac{k}{k_c}\right)\frac{du}{dt} + ku = F_{ext} + \frac{c}{k_c}\frac{dF_{ext}}{dt} \tag{3.3}$$

In the absence of F_{ext} (free vibrations) we have

$$\frac{mc}{k_c}\frac{d^3u}{dt^3} + m\frac{d^2u}{dt^2} + c\left(1 + \frac{k}{k_c}\right)\frac{du}{dt} + ku = 0 \tag{3.4}$$

with four absolute parameters: m, c, k and k_c.

If we define the damping parameters for the SLS-system according to:

$$K(Kelvin) = \frac{km}{c^2} \tag{3.5}$$

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[1] Throughout this article the extensional viscosity will be denoted by η.

and

$$M\,(Maxwell) = \frac{c^2}{k_c m} \tag{3.6}$$

and introduce a dimensionless time variable according to

$$\tau = Dt \qquad \left([D] = \left[t^{-1} \right] \right) \text{ with } D = \frac{c}{m} = c'$$

the differential equation (3.3) can now be rewritten:

$$M\frac{d^3u}{d\tau^3} + \frac{d^2u}{d\tau^2} + (1 + KM)\frac{du}{d\tau} + Ku = 0 \tag{3.7}$$

with only two parameters K and M according to (3.5)and (3.6).

Equation (3.7) reduces simply to the second order DE for the *kcm*-system :

$$\frac{d^2u}{d\tau^2} + \frac{du}{d\tau} + Ku = 0 \tag{3.8}$$

by putting M equal to zero. K is again equal to $\frac{km}{c^2}$. Since the well known expression for the critical value of c: c_{cr}, is $c_{cr} = 2\sqrt{km}$ we find that $\frac{c}{c_{cr}} = \zeta = \frac{1}{\sqrt{4K}}$ and $K = \frac{1}{\sqrt{\zeta^2}}$; ζ being a standard measure for the damping of the *kcm*-system. For another standard measure: the *Quality* (Q-factor), approximately equal to $\frac{1}{2\zeta}$, we have : $Q = \sqrt{K}$.

4. Forced vibrations

The harmonic response of a material is normally presented in terms of the so-called dynamic modulus (ratio of the amplitudes of stress and strain or F_{int} and u terms of equation (3.2)) and the loss angle (phase shift between stress and strain), or its tangent: the loss tangent or loss factor: (ref. 1 and 8). These are material-intrinsic (internal) quantities. For clearness' sake we stress that all expressions figures etc. in this paragraph are *independent* of mass. In other words: these results hold *for any value of a mass* which may be (or may not be) attached to the systems under consideration: SLS and Kelvin. The result for a SLS under forced harmonic vibration is shown in Figure 4.1. It follows easily from (3.1), (3.5) and (3.6) that *the modulus effect* (relative increase of the dynamic modulus when going from very low to very high frequencies) is $\frac{k_c}{k} = \frac{1}{KM}$.[2]

The maximum of $\tan\delta$ is:

$$\frac{t_{cr} - t_{rel}}{2\sqrt{t_{cr} t_{rel}}} = \frac{1}{2\sqrt{KM\,(KM + 1)}} \tag{4.1}$$

occurring at the radial frequency:

$$\omega_{max} = \frac{E}{\eta}\frac{1}{\sqrt{KM\,(KM + 1)}} \tag{4.2}$$

The correspondence between the loss factor $\tan\delta$ and otier damping and time-dependent material parameters is given in Table III.1.

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[2] It is worthwhile to note that the mass included in the definitions of K and M (3.5, 3.6) vanishes in the product KM.

For large values of KM (little damping), $\tan\delta_{max}$ will approximate $\frac{1}{2KM} = \frac{E_\eta}{2E}$ and $\omega_{max} \approx \frac{E_\eta}{\eta}$. This maximum damping frequency is a characteristic frequency (materials property) of a SLS-type material under forced harmonic vibration, and should be distinguished from the frequency of periodic damping of a free damped SLS-system. That frequency is determined by the materials properties and the structural parameters (size, mass) of the vibrating system.
The loss peak will be low and broad in case of low damping, reasonably corresponding with little variation of the loss factor over a wide range of frequencies, as is the case with most solids. This damping behaviour is often termed *solid damping*.
It is noteworthy that the shape of the curves is fully determined by k and k_c (or: by the relaxed and the unrelaxed modulus. Changing c will merely result in a *shift* of the curves along the log ω (!) axis.

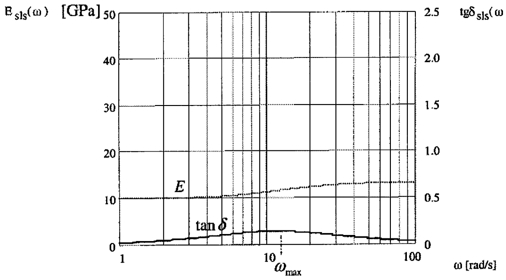

Figure 4.1 Dynamic modulus (dotted curve) and loss factor (solid curve) for a Standard Linear Solid

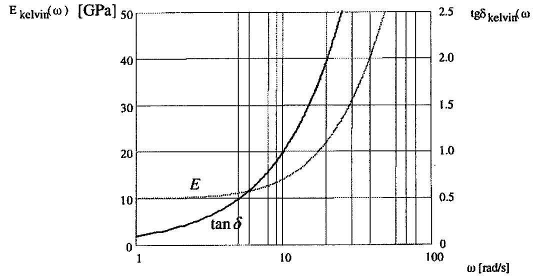

Figure 4.2 Dynamic modulus (dotted curve) and loss factor (solid curve) for a Kelvin - type material

The applied parameters are:

SLS – model:	$E = 10$ GPa	$Kelvin$ – model: $E = 10$ GPa
	$E_\eta = 3.33$ GPa	$\eta = 1$ $GPas$
	$\eta = .25$ $GPas$	

Figure 4.2 shows the dynamic modulus and the phase shift of a Kelvin- system.

The dynamic stiffness of that system is $\sqrt{k^2 + (\omega c)^2}$ and the phase shift between stress and strain . $\delta = arc\tan\frac{\omega c}{k}$. Or in terms of material parameters: $E_{dyn} = \sqrt{E^2 + (\omega\eta)^2}$ and $\delta = arc\tan\frac{\omega\eta}{E}$.

For comparison, the parameters of the SLS- and Kelvin-systems depicted in Figure 4.1 and Figure 4.2 have been chosen in such a way that the creep time and the relaxed modulus are the same for both systems and that the characteristic features of the curves for both systems are clearly depicted. Apart from that the parameter choice is arbitrary and not reflecting particular materials.

It can be seen that in the low-frequency region the dynamic moduli of both systems follow each other fairly closely. Beyond the maximum of the loss factor (SLS-system) the Kelvin-model predicts a dynamic modulus and a $\tan\delta$ running up towards infinity for high frequencies (phase shift between stress and strain rising to $\frac{\pi}{2}$) indicating total dissipation of the work exerted upon the system.
This deviates from the actual behaviour of a large majority of building materials which possess very – or fairly low values of δ, remaining fairly constant over a wide frequency range (see references cited on p.2), and going down towards zero for both very high and very low frequencies (p.3). The actual dynamic modulus may exhibit a substantial increase ("cross-over") over a certain frequency range (the modulus effect) but will then level off to a fairly constant value (Figure 4.1; ref. 1, 4 and 8 include many actual examples). It will never rise towards infinity.
We remind once more that these results hold for material-intrinsic (constitutive) behaviour. In engineering practice, attention will often focus upon external quantities: displacements versus externally applied force (Figure 3.1). Vrouwenvelder (ref. 9) has pointed out that the (mass-dependent) ratio of the amplitudes of displacement and external force is much less sensitive to the rheological description of the material (in our case Kelvin or SLS) than the (mass-independent) stress/strain dependency within the material. We will return to this in § 6.

5. Free vibrations

5.1 *General form of free-damped vibrations*

Starting again with Equation (3.7) and substituting $u = u_o e^{r\tau}$, we obtain the characteristic equation:

$$Mr^3 + r^2 + (1+KM)r + K = 0 \tag{5.1}$$

It is well known that a third order algebraic equation possesses either
- three real roots r_1, r_1, r_3 or
- two complex conjugate roots $r_{1,2} = -(a \pm ib)$ and one real root $r_3 = -c$.[3]

We can write the solutions as:

$$A_1 e^{-(a\pm ib)\tau} + A_2 e^{-c\tau} \tag{5.2}$$

or

$$A_1 e^{-a\tau} \cdot \cos(b\tau) + A_2 e^{-c\tau} \tag{5.3}$$

in which a and c are positive. In the case of periodic solutions b is the radial frequency (referred to the dimensionless time variable τ).
The actual radial frequency is $\omega = bD = b\frac{c}{m}$ (§3).

[3] The symbol c here for one of the roots should not be confused with the damping coefficient in a rheological model

We furthermore note that the logarithmic decrement (*LD*) of the periodic vibrations is given by $LD = 2\pi\frac{a}{b}$.

Expressions (5.2) and (5.3) present the general form of the free damped vibration of the SLS-system. We can distinguish the following damping types:

(1) $x = A_1e^{r_1\tau} + A_2e^{r_2\tau} + A_3e^{r_3\tau}$

$(r_1, r_2, r_3$ *real negative), fully exponential damping (EXP)*

(2) $x = A_1e^{-a\tau}\cos(b\tau) + A_2e^{-c\tau}$

(a, b, c, real positive), mixed periodic damping and exponential damping (EXPER)

In the latter case we have the subdivision:

(1) $a > c$ with the periodic term damping out more rapidly than the exponential term (EXPer)

(2) $a < c$ with the exponential term vanishing more rapidly than the periodic term (exPER)

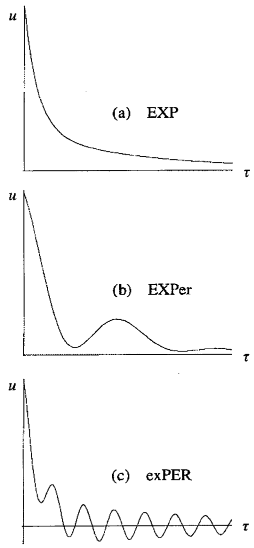

Figure 5.1 Damping modes of SLS-type materials

a) EXP; fully exponential damping

b) EXPer; mixed periodic and exponential damping with the periodic term damping out more rapidly

c) ExPER; mixed periodic and exponential damping with the exponential term damping out more rapidly

Figure 5.1 illustrates these vibration modes.

In an article written as early as 1967, Struik (ref. 9) showed that the general form if the free damped vibrations of a linear material is either:

- a superposition of (negative) exponentials (non-periodic damping), or:
- the sum of one exponentially damped harmonic vibration and a number of exponentials (periodic damping).

(The contents of his paper do not seem to have found their way towards the engineering world.)

Clearly, the solution types (1) (EXP) and (2) (EXPER) correspond with Struik's general solution types. The only distinction with our SLS damping modes is that there can be any number of exponentials in his general result. The distinct modes we have introduced: EXP, EXPER (with subdivision EXPer and exPER), can be likewise distinguished in Struik's general result. In other words: *the distinction EXP, EXPER (EXPer, exPER) holds quite generally for the free vibrations of linear solids.*
Damping similar to exPER is sometimes seen by experimentators and then attributed to "dynamic effects" or "parasitic impedances" in the measuring set-up. It follows from the present analysis and Struik's general result (ref.10) that these researchers may well have seen the actual damping behaviour of the material under analysis.
On the other hand there are many cases where the "extra" damping phenomena of the SLS-description (Figure 5.1b and c) are too insignificant to show up. This is obviously the case when *c* (5.3) is either very small or very large compared with a. Since it can be shown (Appendix I) that the value of c/a approximates *2KM* for larger values of *K* and *M* we can conclude that these extra damping features will gradually vanish for larger values of *K* and *M*, that is: for materials with little damping (§ 5.2., Appendices I and II).

For *kcm* systems we have the well-known results:

$$u = A_1 e^{-a\tau}\cos(b\tau) \quad \text{(exponentially damped harmonic)} \quad for\ K > \tfrac{1}{4} \tag{5.4}$$

$$u = A_1 e^{-r_1\tau} + A_2 e^{-r_2\tau} \quad \text{(two negative exponentials)} \quad for\ K < \tfrac{1}{4} \tag{5.5}$$

with: $a = \frac{1}{2}$, $b = \sqrt{4K-1}$, $\tau = \frac{c}{m}$, $LD = \dfrac{2\pi a}{b} = \dfrac{\pi}{\sqrt{4K-1}}$

For low damping (*K* large) *LD* will approximate $\dfrac{\pi}{\sqrt{4K}}$. In terms of *k, c* and *m* we have the well-known expressions:

$$LD = \frac{\pi c}{\sqrt{mk}} = \frac{\pi c}{m\omega} = 2\pi\zeta\ \text{, with } \zeta = \frac{c}{c_{cr}} = \frac{1}{\sqrt{4K}} \text{ and } \omega = \sqrt{\frac{k}{m}}\ . \tag{5.6}$$

5.2 *KM-diagram*

We have solved the third order Equation (4.1) for various intervals and values of the parameter-pair (*K,M*), using a programmable scientific calculator[4]) (Hewlett-Packard: HP48GX). A mathematical software package for a personal computer can alternatively be employed to obtain the solutions.

[4] All symbolic derivations (computer algebra), necessary for this article, were also carried out on the HP48GX.

When these results are plotted in a *KM*-diagram (Figure 5.1), different domains can be identified in the *KM*-space, corresponding with different types of root-combinations (complex conjugate and/or real (§4)), representing the damping modes EXP, EXPer and exPER of the SLS-system as distinguished in the previous paragraph. Solution types of the *kcm* - system are also represented in the diagram.

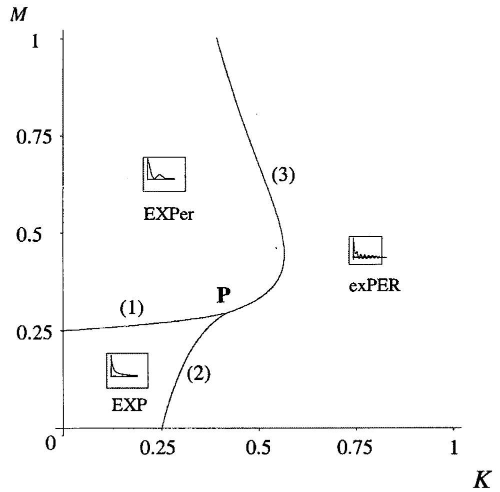

Figure 5.2 *KM-diagram in which the EXP, EXPer and exPER domains are marked*

These domains are separated by the curves (1), (2) and (3) for which the equations are given in Appendix I. Curves (1) and (2) enclose the domain of fully exponential damping (EXP), corresponding with three (different) real roots. Together, curves (1) and (2) constitute a twodimensional generalization of the critical damping point of the *kcm* system and can be termed *the critical damping curve* for the SLS system. Two of the roots will be equal on (1) and (2). Curves (1) and (2) have one point in common (P), corresponding with three equal roots:

$$r_1 = r_2 = r_3 = -\frac{9}{8}$$

The coordinates of the special point P are:

$K = \frac{27}{64}$, $M = \frac{8}{27}$, as will be shown in Appendix I.

Beyond the (1)-(2) domain, there will be mixed periodic and exponential damping (EXPER). Curve (3) separates the area where exponential damping will be dominant (EXPer, left hand side of (3)) from the essentially periodic damping domain (exPER) on the right hand side of curve (3). The further we move away from the (1)-(2) domain (towards higher values of *K* and/or *M*) the lower the damping will be.
The *K*-axis and the *M*-axis correspond with the KELVIN system and the MAXWELL-system respectively. Coordinates ($\frac{1}{4}$,0) and (0,$\frac{1}{4}$) correspond with critical damping of these systems; for lower val-

ues of K and M along the axes there will be (simple) exponential damping; for K (or M) > $\frac{1}{4}$, the damping will be periodic.

Ideal elastic materials (no damping) correspond with K and/or M infinite.

Purely viscous materials (Newtonian liquids) are represented by the origin of the diagram ($K = M = 0$).

Many further aspects of the diagram are treated more fully in Appendix I.

6. Discussion

6.1 *Forced and free vibrations of the kcm system*

In § 4 it was shown that in the case of *forced vibrations* the frequency dependency of the dynamic modulus and the loss factor $\tan\delta$ of the Kelvin-system deviates substantially from the dynamic response of materials, particularly in the high-frequency region where the predicted stiffness is far to high.

This discrepancy would tend to under-estimate the predicted dynamic displacements of systems. The practical effect of this on structures will often be rather limited, firstly because dynamic response will be small anyway for frequencies appreciably higher than the resonance frequencies, and more particularly because – as cited (ref.9) in § 4 – the dynamic characteristics of the Kelvin and SLS systems with a mass attached differ much less than the "internal" quantities: dynamic modulus and loss factor, of both systems. In this case the Kelvin-description has the advantage of greater simplicity of the pertinent expressions.

There are of course other situations in civil and building research and practice where a full assessment of the time-dependent behaviour of the material is essential for obtaining reliable results. Examples are: dynamic stiffness contribution of layered components in road constructions or in composites, propagation and attenuation of mechanical waves in media, force and motion transmissability of damping systems.

In the case of *free vibrations* the (periodic) solution of its differential equation (3.8), or in unreduced form: $m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0$ is the product of an exponential and a harmonic function (5.4).

Although this solution does not conform with the general form of a free damped periodic vibration: the sum of such an exponentially decaying harmonic function *and* one ore more exponential functions (Struik, ref. 10), this will not be a serious discrepency in practice because it is the damped harmonic component that will be of main interest. Furthermore, we have seen (§ 5) that the extra features of the general solution tend to diminish when going from high- to low-damping materials.

A complication with the (unmodified) *kcm*-system is the following. If we accept the solution taken from the DE of the *kcm*-system as a possibly good approximation in the case of low damping, we have the result for the logarithmic decrement LD approximately equal to $LD = \frac{2\pi c}{c_{cr}}$ (5.6) with $c_{cr} = \sqrt{4km}$.

It is well known that a problem arises here: if we would decrease the resonance frequency, say by a factor 2, by increasing the mass coupled with the system with a factor 4, c_{cr} would increase by a factor 2 and *LD* would decrease by a factor 2 (ref. 2). However the logarithmic decrement of most materials has the tendency to remain fairly constant over a wide range of frequencies. Here the outcome is in marked contrast with reality. We will return to this in section 6.2 to see how this complication is resolved.

Our findings with both free and forced vibrations on the (unmodified) *kcm* system can also be summarized by stating that the "viscous friction force" $c\frac{du}{dt}$ (or $\eta\frac{d\varepsilon}{dt}$) in the DE for the system, overestimates the rate (and frequency) dependency of that force in most materials.

6.2 *Modifications of the kcm description*

There have been numerous attempts with the kcm-system to modify the DE, and its outcome, in order to obtain better correspondence with material behaviour under both forced and free vibrations.

In the case of *forced vibrations,* worked out in §4, one can introduce a frequency dependency of *c*. If we assume – following Beards (ref. 3, p. 25) and many other authors – that *c* and η are inversely proportional with ω over a certain frequency range, then δ and *E* would be constant over that frequency range. This is better in agreement with experiment (ref. 1, 2, 3 and 8), particularly for δ. The result for *E* is less satisfactory because of the lack of a modulus effect. This is a principal difficulty, because – as Kramers and Kronig have shown – a non-zero value of δ over a certain frequency range should always be accompanied by a rise of the dynamic modulus and vice versa (Kramers/Kronig relations, see ref 11). In fact tan δ should roughly follow the derivative of $E(\omega)$. (Figure 4.1 shows this quite well). Absence of a predicted modulus effect is of course not a practical difficulty in cases where the modulus effect is very small. However, many fairly low-damping construction materials (concrete, wood, engineering plastics, asphalt) show a significant modulus-effect.

Let us apply the assumption $\left(c \triangleq \frac{1}{\omega}\right)$ also to the case of *free damped vibrations* of the *kcm* system. We can follow the standard results (5.4), (5.6) for the (unmodified) *kcm* system and substitute $\left(c \triangleq \frac{1}{\omega}\right)$ under the assumption that ω can be treated as a constant when solving the DE for *t*. Strictly speaking this is never the case for damped vibrations, because any damped harmonic function has a certain frequency range. But it is a good approximation in case of *low damping* because then the ω(Fourier) spectrum of the solution function will be narrow. With *c* inversely proportional with ω, *LD* and ς will be frequency-independent, in fair agreement with most structural materials. The more general shape of damped vibrations as given by Struik and shown in this article (Fig. 5.1) and discussed in § 6.2, requires a description in terms of a third order DE (3.7) and will not appear in this approximation.

In case of *high damping* an approximation which follows the standard *kcm*-solution, with *c* a function of frequency, breaks down. A numeric Fourier transformation analysis would be required to establish the nature (periodic, non-periodic) and the magnitude of the time-dependent displacements. Adoption of a three-parameter model description – SLS – would seem to be a far more practical approach in this case.

6.3 *The SLS solutions. Solid damping*

As shown in §5.1, the SLS - system presents the general form of *free damped* vibrations. As such it is an adequate model for describing and interpreting damped vibrations in a variety of materials including both the regular low damping structural materials, and moderately (asphalt, soils) as well as high-damping materials (elastomers, foams, fresh concrete). Examples and an application are given in Appendix IV .

The solution for *forced* vibrations is also in good agreement with experiment (ref. 1), particularly for the dynamic modulus (modulus effect). The $\tan\delta - \omega$ curve differs from the fairly constant- δ behaviour characteristic for many low-damping solids. This discrepancy follows from the fact that the SLS solid possesses one relaxation mechanism instead of many: the actual situation in many solids. The relaxation spectra of each of these mechanisms join up to form a on-going $tan(\delta) - \omega$ curve which may be fairly flat (Figure 6.) This is the origin of the "solid damping" of most materials.

The "cross over" of the dynamic modulus will be less steep and will extend over a larger frequency-range than in the SLS-curve for the modulus effect. These discrepancies will hardly affect the applicability of the SLS model for general engineering analysis of vibrating structures and prediction of vibration phenomena.

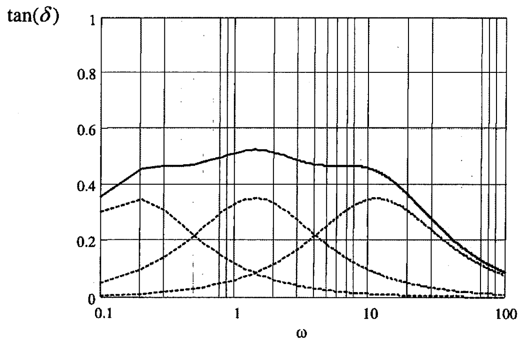

Figure 6.1 Spectra of three relaxation mechanisms, joining up to form an on-going $tan(\delta) - \omega$ curve (schematically)

7. Conclusion

Once a Standard-Linear-Solid type description of a material is obtained (with for instance parameters E, E_η and η (extensional viscosity)), a dynamic model can often be set up for a specimen or a structural component of that material according to Figure 1.1, with mechanical parameters k, k_c, c and m (concentrated mass). We have presented the third order differential equation for such a system and we have shown that its free damping modes are the general free damping modes of vibrating materials. The shape and the parameters of its damping curves are related to the material properties and dimensions in a simple manner. Straight-forward relations can be set up for this model between vibration behaviour and other time dependent phenomena such as creep and stress relaxation.

Of course, as a three-parameter model the SLS model is still an approximation. A full visco-elastic treatment, such as presented by Struik, or on the basis of a generalised Maxwell-chain model (4), may be required for some purposes. Such treatments, however, are rather elaborate, both experimentally and mathematically, and stand little chance to be adopted in general engineering practice.

Whereas the SLS description is shown to be more general and more versatile in handling damping situations in materials and – systems (composites), the standard KELVIN (*kcm*) model retains its applicability for low-damping materials and constructions.

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APPENDIX I

Some mathematical aspects of the K,M-equation and the K,M-diagram

- *Relating the damping modes with the ratio* $\frac{E_\eta}{E}$.

Since $KM = \frac{k}{k_c}$ (as follows from Equations (3.5) and (3.6)), a SLS-type material with particular values of its moduli E and E_η, is represented in the *KM*-diagram by a orthogonal hyperbola KM = C(onstant).
(Three of these hyperbolas have been drawn in the *KM*-diagram shown in Appendix I, Figure I.1).
The viscosity (η), and the mass associated with the vibration, will determine the position on the hyperbola.
It follows from the value of the product of the coordinates K and M at point P: $KM = \frac{1}{8}$ that these hyperbolas will only intersect curves (1) and (2) if $C > \frac{1}{8}$. So a SLS material can only enter into fully exponential damping mode (EXP) if $E_\eta > 8E$. For $E_\eta < 8E$ the damping will be according to one of the harmonic modes EXPer or exPER.
Furthermore, we will see in this appendix that the hyperbolas will $KM = C$ only intersect curve (3) if $C < \frac{1}{2}$ ($E_\eta > 2E$). This means – in combination with the preceding result $E_\eta > 8E$ – that a SLS system can only enter into EXPer mode if E_η is in the interval $2E < E_\eta < 8E$. For ratio's smaller than 2, exPER is the only possible damping mode of the SLS system.

- *Relating the roots with K and M*

A third degree equation with roots r_1, r_2, r_3 can be written as:

$$(r - r_1)(r - r_2)(r - r_3) = 0$$

or:

$$r_3 - (r_1 + r_2 + r_3)\, r^2 + (r_1 r_2 + r_2 r_3 + r_3 r_1) r - r_1 r_2 r_3 = 0$$

If we compare this with:

$$Mr^3 + r^2 + (1+KM)r + K = 0 \text{ (Equation 3.7)}$$

we find:

$$-(r_1 + r_2 + r_3) = \frac{1}{M},\; r_1r_2 + r_2r_3 + r_3r_1 = \frac{1}{M} + K \text{ and } -r_1r_2r_3 = \frac{K}{M},$$

or, with $r_{1,2} = -(a \pm ib)$ and $-r_3 = c$:

$$2a + c = \frac{1}{M} \quad \text{(I.1a)}$$

$$a^2 + b^2 + 2ac = \frac{1}{M} + K \quad \text{(I.1b)}$$

$$(a^2 + b^2)c = \frac{K}{M} \quad \text{(I.1c)}$$

- *Equations for the curves (1), (2), (3) separating the vibration mode domains in the K,M diagram.*

We know that the condition $a = c$ separates the EXPer- and exPER-modes ($a > c$ and $a < c$, respectively), and that the periodic and exponential modes (EXPER and EXP) are separated by the condition $b = 0$.
Putting $a = c$, and eliminating a and c in Equations (I.1), we obtain:

$$KM^2 - \frac{1}{2}M + \frac{1}{9} = 0 \qquad \text{(I.2)}$$

Curve (3) (Figures 5.1 and I.1) is the graphical representation of Equation (I.2).
The set of Equations (I.2) and $KM = C$ (§5) has real common roots (K,M) only if $C < \frac{1}{2}$.
This means that a SLS-material can only enter in EXPer-vibration mode if $E_\eta > 2E$ (Figure 1.1).
Some hyperbolas $KM = C$ with different values of C have been drawn in diagram I.1

If, on the other hand, we put $b = 0$, one obtains, after eliminating a and c in Equation (I.1):

$$-1 + 4K - 20KM + 8K^2M^2 + 4M(1 + KM)^3 = 0 \qquad \text{(I.3)}$$

Curves (1) and (2) are branches of the graphical representation of Equation (I.3).

If we put: $b = 0$ and $a = c$ (three equal roots), the coordinates of the point P, equaling: $K = \frac{27}{64}$ and $M = \frac{8}{27}$, can be readily deduced from the Equations (I.1a, I.1b, I.1c).
We note that in this case $KM = \frac{1}{8}$.
Curve (4) connects the positions on the hyperbolas KM=C(onstant) where the logarithmic decrement is at a maximum.
The curve indicates the area where – given a certain material (E, E_η) – maximum damping of the periodic oscillations can be obtained (by adjusting, for instance, the mass associated with the vibration). The maximum is rather flat for higher values of C (= KM), but very pronounced in the EXPER-area close to the special position (P). The equation for curve (4) will be given in Appendix II.

- *Approximate solutions for a, b and c.*

An excellent approximation in case of low or moderate damping ($KM > 5$)is:

$$a = \frac{1}{2}\cdot\frac{1}{KM^2 + M + 1}; \qquad b = \sqrt{\left(K + \frac{1}{M}\cdot\left(1 - \frac{KM^2 + \frac{M}{4} + 1}{\left(KM^2 + M + 1\right)^2}\right)\right)}; \qquad c = \frac{1}{M}\cdot\frac{KM^2 + 1}{KM^2 + M + 1}.$$

Using the expressions for a and b we can obtain the following approximation for the logarithmic decrement:

$$LD = \frac{\pi}{KM}\cdot\frac{1}{\sqrt{KM^2 + 3M + 2}}$$

indicating low damping for higher values of KM. (Exact) solutions for a, b and LD, holding for the KELVIN-system, were given in §5.
Using the expressions for a and c we also obtain the approximation: c/a = 2KM for higher values of K and M. The ratio c/a is of major influence on the shape of the damping curves (§5).

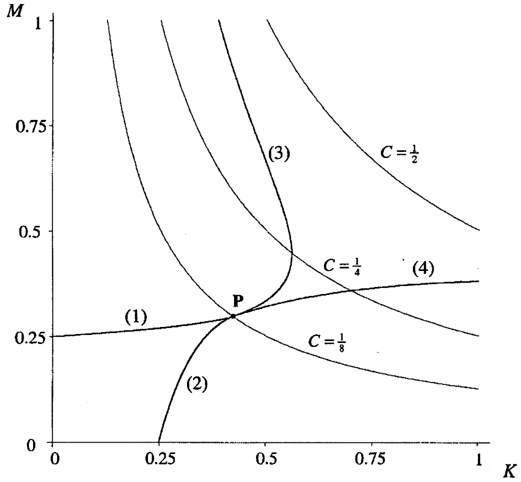

Figure I.1 K,M-diagram, showing the EXP-, EXPer- and exPER-domains and the curves (1), (2) and (3) separating these domains.

Three hyperbolas KM = C, with C = $\frac{1}{8}$, C = $\frac{1}{4}$ and C = $\frac{1}{2}$, corresponding with materials with creep coefficients 8, 4 and 2 respectively, (see Appendix III) have been drawn.

Curve (4) connects the positions on the hyperbolas KM = C where the periodic damping is at a maximum.

APPENDIX II

Logarithmic decrement, particularly in the low damping area.

The SLS description proposed in this article interprets both very high (non periodic) damping, such as in viscous or viscoplastic materials like fresh concrete and soft soils, and harmonic damping with any value of the logarithmic decrement. In dealing with building materials periodic damping will be prominent; particularly materials for structural purposes are essentially elastic in the vibration domain, with moderate or low damping of free vibrations excited in such materials.

We will therefore study the periodic damping in more detail in this Appendix, with special attention to the logarithmic decrement (*LD*). Results will again be presented in the *KM*-diagram, with double logarithmic representation more practical when a wide range of *K* and *M* values is to be surveyed.

We will first present a generalisation of curves (1) and (2). In §5, these curves were shown to separate the area of the *KM*-diagram where the damping is fully non-periodic (the EXP area) and the remaining area with periodic damping (EXPer or exPER). This means that the periodic damping is at a maximum on these curves, corresponding with $LD = \infty (b = 0)$. The expression for curves for any other value of *LD* can be obtained from the expressions (I.1) after extensive algebraic manipulation. The expression is:

$$(w-3)\left(K+M(1+KM)^3\right)+(2w+3)(1+KM)^2-\left(\frac{(w+3)}{2}+\frac{(5w-3)}{(w+1)}KM\right)^2=0 \qquad \text{(II.1)}$$

in which $w=\dfrac{4\pi^2}{(LD)^2}$.

Equation (II.1) is not valid for w = 3. In that case we have:

$$K+M(1+KM)^3-(1+KM)^2=0 \qquad \text{(II.1a)}$$

One can easily check that Equation (II.1) reduces to Equation (I.3) for $w = 0$ $(LD = \infty)$.
Curves for constant *LD* representing Equation II.1 are drawn in Figures II.1 and II.2, using linear and logarithmic axes respectively.

In the logarithmic *KM*-diagram (Figure II.2) we can again study the behaviour of a SLS-system (E, E_η fixed) of which the damping is altered by either changing the viscosity itself or the mass associated with the vibration. We have already noted that for such a system $KM = E/E_\eta$ is a constant, meaning that upon alteration of *c* and/or *m* the system moves along a hyperbola $KM = C$ (or a straight line log *K* + log *M* = log *C* in a logarithmic presentation). We note now that for any specific *LD* curve, there will be a line *KM* = *C*, with a specific value of *C*, contacting the *LD*-curve. The point of contact is at

$$K=\sqrt{C(1+C)^3},\quad M=\sqrt{C(1+C)^{-3}} \qquad \text{(II.2)}$$

and this is the position of the maximum of *LD*, mentioned already in Appendix I, when the material moves along the line $KM = C$.
The positions of these *LD*-maxima for different *C*-values are located on curve (4) with equation:

$$M(1+KM)^3 - K = 0 \qquad \text{(II.3)}$$

For high values of *K* this equation is approximated by $M = \dfrac{1}{\sqrt{K}}$.

In special position (P) – which is also on curve (4) – the periodic component will vanish and *LD* will be infinite.

The expressions for Equations II.2 and II.3 as well as the co-ordinates *K* and *M* of the point of contact can be obtained fairly easily when the intersections of curves (II.1) and hyperbolas $KM = C$ are studied algebraically. These (two) intersections should coalesce at the positions of the maximum points, from which a relation between *w* and *c* can be obtained.

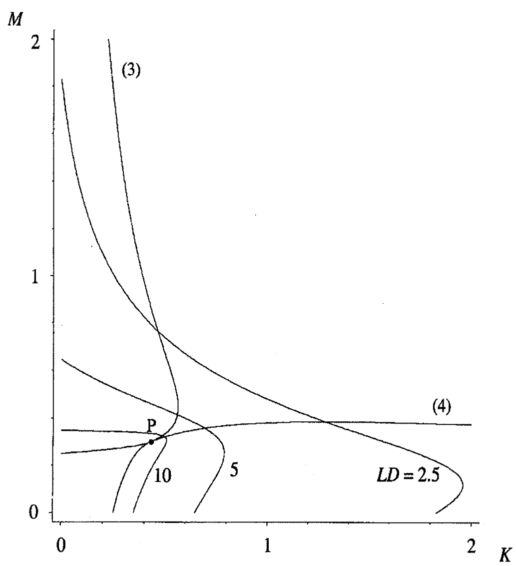

Figure II.1 K,M-diagram.
Constant LD (Logarithmic Decrement) curves have been drawn for the LD-values indicated: 2.5, 5, 10 and ∞. *Curve (4) connects the maxima of LD on the hyperbolas KM = C (shown in Figure I.1, not shown here).*

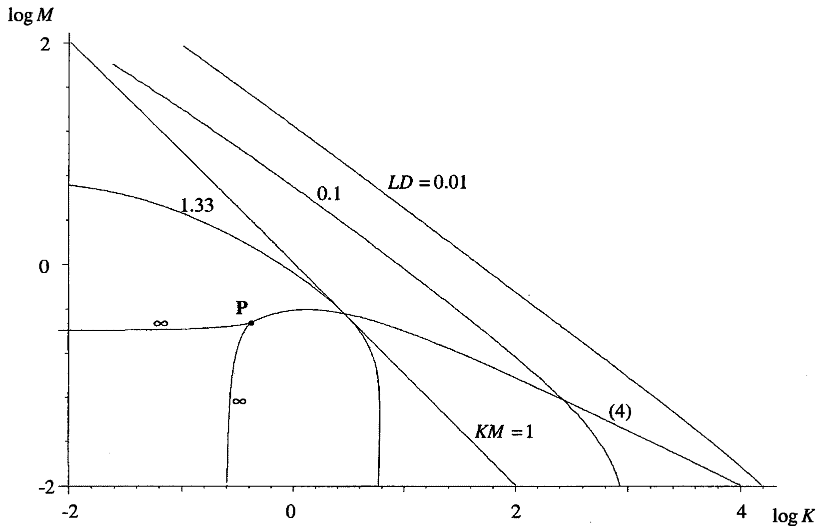

Figure II.2 K,M-diagram with logarithmic K- and M-axis. LD-curves have been drawn for values indicated in the diagram.
As an example the LD-curve for 1.33 touching line KM=1 is shown. The point of contact is at position $K = 2\sqrt{2}$ *and* $M = \frac{1}{4}\sqrt{2}$ *(using Equation II.2).*
LD = 1.33 (Table II.1) is the maximum value of LD on the line KM = 1.

Appendix III

Interrelationships between various parameters

It is well known that creep and stress-relaxation of a SLS-type material under uniaxial load are determined by a short term ("unrelaxed") modulus, E_{unrel}, a long term ("relaxed") modulus, E_{rel}, and an extensional viscosity , determining the creep rate and the relaxation rate. The correspondence with the parameters of the mechanical model introduced in §1 is:

$$k = \frac{S}{\ell} E_{rel}\,; \quad k_c = \frac{S}{\ell}\left(E_{unrel} - E_{rel}\right); \quad c = \frac{S}{\ell}\eta \qquad \text{(III.1)}$$

with $E_{rel} = E$ and $E_{unrel} - E_{rel} = E_\eta$, in which S and ℓ are the cross-section and the length of a rod-shaped specimen of the particular material under load. It follows from Equation (III.1) that the ratio's between k, k_c, and c are independent of the dimensions (S,ℓ) of the specimen.

The uniaxial constitutive equation for a Standard Linear Solid is:

$$\sigma + t_{rel}\frac{d\sigma}{dt} = E\left(\varepsilon + t_{cr}\frac{d\varepsilon}{dt}\right) \tag{III.2}$$

with t_{cr} the creep time, and t_{rel} the relaxation time, according to:

$$t_{cr} = \eta\left(\frac{1}{E_{unrel} - E_{rel}} + \frac{1}{E_{rel}}\right) = \frac{c}{k_c} + \frac{c}{k} \tag{III.2a}$$

$$t_{rel} = \eta\frac{1}{E_{unrel} - E_{rel}} = \frac{c}{k_c} \tag{III.2b}$$

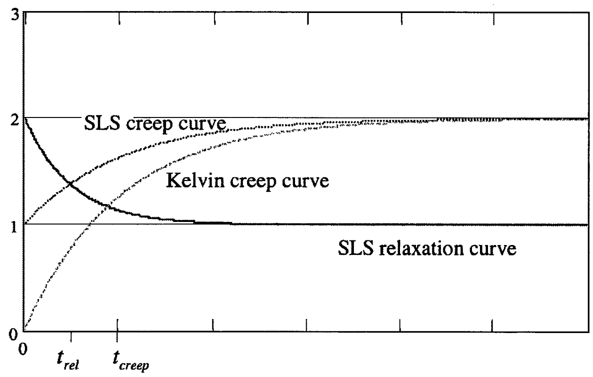

Figure III.1 Creep curve and stress relaxation curve for a SLS-type material with E =E$_\eta$. The creep curve of a Kelvin-type material with the same relaxed modulus and the same creep time is also shown.

Creep and stress relaxation are shown schematically in Figure III.1 for a material with a creep coefficient (φ) equal to 1.

Since: $\varphi = \frac{\varepsilon_{cr}}{\varepsilon_{el}} = \frac{k_c}{k} = \frac{1}{KM}$

we find that hyperbolas *KM* = constant in the *KM* diagram correspond with materials with a particular value of their creep coefficient.

The above-given summary of the deformation - and dynamic properties of a SLS material shows that the essential quantities: creep coefficient, relaxation modulus, loss angle, etc., are all very simply related to the parameters *K* and *M* which we have introduced in our treatment of the free damping behaviour.

It can be seen from Figure III.1 that a KELVIN type material has a creep coefficient which is infinite. The unrelaxed creep modulus and relaxation modulus are also infinite. This demonstrates once more that the KELVIN model fails to simulate actual time dependent material behaviour.

Table III.1 Corresponding values of the creep coefficient ($1/C$) and the maximum periodic damping (LD_{max}) of a SLS. The characteristic properties of a material under forced vibrations (Appendix III): modulus effect and $(\delta)_{max}$, are also included in the table.

$\frac{1}{C}$	C	LD_{max}	$\tan(\delta)_{max}$
$1/C=\varphi$ creep coefficient, also: modulus effect	$C=\frac{k}{k_c}=\frac{E}{E_\eta}=KM$	Maximum values of logarithmic decrement for corresponding $\frac{k}{k_c}$-values	δ is phase shift between stress and strain under harmonic load
.001	1000	0.00157	0.0005
.003	333	0.00471	0.0015
.01	100	0.0157	0.005
.03	33	0.0468	0.0148
.1	10	0.153	0.0477
.3	3.3	0.442	0.1316
1	1	1.33	0.3535
3	.33	3.63	0.75
5	.2	6.61	1.02
8	.125	∞	1.33

Table III.1 lists a number of *C*- and corresponding *LD*-values. In this appendix we have shown that the value of C (= *KM*) of a Standard Linear Solid measures the creep, stress relaxation and modulus effect of a SLS. We have seen that $1/C$ is the creep coefficient. So, Table III.1 connects the maximum damping (maximum *LD*) of a SLS material with its creep factor, and we see that the SLS-description allows a rational connection between these different time-dependent material properties.
It should be reminded that these connections have been derived for the SLS-*model*. Materials may have several relaxation mechanisms (§ 2) over a wide time-domain. As a consequence a relation between for instance creep and free damping behaviour will only hold in a particular frequency (or: time) region, as explained at the end of § 2.

APPENDIX IV

Some examples

IV-1 Typical examples of SLS-representations for steel, concrete, and organic materials.
E-values in GPa , η in GPas.

The data in the table below are of an indicative nature. It should be reminded that a SLS-description of a vibrating system includes – in addition to material properties – its dimensions and the mass associated with it. With the exception of *E* and E_η, its parameters are generally not intrinsic material properties. Moreover the material properties related to damping may also be dimension-dependent (§2).

E is *the relaxed (static) modulus, E + E_η is the unrelaxed modulus.* The $\tan\delta$ data were taken from various literature sources. E_η follows from $\tan\delta$ using Equation (4.1)
Data for η in the table were set – rather arbitrarely – in such a manner that the "flat" maximum of $\tan\delta$ is around 10 c/s.
The *LD*-value will of course depend on the mass attached to the system. LD_{max} however, can be estimated using the expressions given in Appendix II. For very low damping we have:

$$\frac{LD_{max}}{\tan\delta_{max}} = \pi \cdot$$

	E	E_η	*KM*	η	$\tan(\delta)$	LD_{max}
Steel	210	0.15	1400	0.0024	0.0003	0.001
Concrete	30	1.2	25	0.019	0.02	0.06
Wood	12	1.5	8	0.024	0.06	0.2
PMMA	3	0.6	5	0.001	0.1	0.3
Cork	0.03	0.01	3	0.0002	0.15	0.5

A good procedure for setting up a SLS description for a unknown material system would be to measure the dynamic modulus and loss factor as a function of frequency (ref. 8) in order to determine *E*, E_η and $\tan(\delta) - \omega$ the dependency. η could then be adjusted to "fit" this dependency for the frequency region under consideration.
If a measured or published value of $\tan\delta$ is fairly constant, it seems best to set $\tan\delta_{max}$ in the middle of the frequency region of interest. This ensures that discrepencies for $\tan\delta$ and *LD* for other frequencies will be minimal.

IV-2 Worked-out example for a bituminous composite

Let us consider a bituminous material for which the deformation behaviour is approximated by the MAXWELL-model, with E = 3GPa, and with extensional viscosity $\eta = 1$ *GPas* at room temperature. We wish to study the longitudinal free vibrations of a specimen (Figure IV.1) with length ℓ = 1m and cross-section : S = 0.01 m^2:

a) of the material as such;

b) of the material reinforced (stiffened and strengthened) with 1% (by volume, V = 0.01) of continuous steel wiring in the longitudinal direction.

For simplicity, we will approximate the dynamics in both cases by concentrating the mass (m) of the material at the free end of the specimen (the other end clamped), in accordance with Figure 1.1. However, in order to obtain agreement of the vibration frequency of the concentrated mass model with the fundamental mode frequency of our beam-shaped continuous specimen, the value of the concentrated mass should be set at $m_{discr} = \frac{4}{\pi^2} m$ rather than m. (The above mentioned frequencies will then be exactly the same in the purily elastic case.)
Lateral effects (Poisson contraction) will be neglected. The only damping mechanism considered will be the viscosity of the bituminous material.

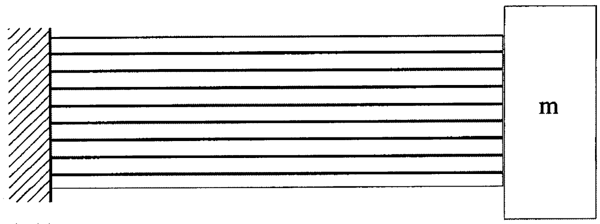

Figure IV.1 Specimen of a bituminous material reinforced with steel wiring.

Clearly the composite material (b) is a standard linear solid, with:

$$\begin{aligned}
k &= 210\frac{S}{\ell}V \text{ GN/m} &&= 21.0\cdot10^6 \text{ N/m}\\
k_c &= 3\frac{S}{\ell}(1-V) \text{ GN/m} &&= 29.7\cdot10^6 \text{ N/m}\\
c &= 1\frac{S}{\ell}(1-V) \text{ GNs/m} &&= 9.7\cdot10^6 \text{ Ns/m}\\
m &= \left(7800V+1000(1-V)\right)S\ell &&= 10.68 \text{ kg}\\
m_{discr} &= &&= 4.33 \text{ kg}
\end{aligned}$$

In the expressions given above we use the (approximate) values for the moduli and the specific masses of the composing materials.

Following the definitions (3.5) and (3.6) of our parameters K and M, we find that $K = 0.927\cdot10^{-6}$ and $M = 0.762\cdot10^{6}$.
It follows from §5 and Figure 5.1 that system (b) is in an EXPER-mode, with one real and two complex conjugate roots of the characteristic equation.
The values of a, b and c obtained are: $a = 0.384\cdot10^{-6}$, $b = 1.496\cdot10^{-3}$, $c = 0.543\cdot10^{-6}$.
The logarithmic decrement of the vibrations will accordingly be 0.00162, indicating very low damping.

Using $D = \frac{c}{m}$ (§3), the damping frequency f ($f = \frac{\omega}{2\pi}$, ω is radial frequency) is found to be 544 c/s.

Figure IV.2 illustrates the damping behaviour.

Case (a) corresponds with a simple MAXWELL system with $M = 0.822 \cdot 10^6$. Since M (highly) exceeds the value for critical damping ($\frac{1}{4}$), there will be periodic damping, with $LD = 0.0035$ and with frequency 405 c/s.

We will elaborate somewhat on the second example (system (b)) by lowering the viscosity of the reinforced bituminous material (corresponding with higher temperatures). Similar extensions in the K,M domain could be made by varying the length of the specimen, or the volume fraction of the steel reinforcement.

We first note that $KM = \frac{k}{k_c} = 0.707$, independent of viscosity and temperature. This means that

the system will move along the hyperbola $KM = 0.707$ in the K,M-diagram when the viscosity is altered. Since KM exceeds $\frac{1}{8}$, this system will never be in the fully exponential damping mode EXP (Appendix I).

Furthermore, since 0.707 > 0.5 (0.5 is the upper limit value of KM for EXPer-damping, Appendix I), the material will vibrate according to the exPER damping mode and there will be no transition from exPER to EXPer damping. (We would have had such a transition if the reinforcement percentage would have been $\frac{1}{2}$ % instead of 1 %).

At the special position $K = M$ ($c = 10399$, $\eta = 1.05$ MPas), we obtain $LD = 1.477$ and $f = 428$ c/s.

On lowering the viscosity slightly further, a maximum of the logarithmic decrement can be obtained (maximum damping of the periodic component). Using expressions (II.2) we find that the maximum is at $K = 1.875$, $M = 0.377$. The maximum is 1.81, which means very rapid damping of the periodic component. The corresponding frequency is 419 c/s.

At still lower viscosities (kPas-range) damping will be very low again, with LD of the order of 0.003.

Table IV.1 lists the vibration and damping parameters of the material for viscosity values ranging from 1 GPas to 1 kPas.

Figures IV.2 illustrate the damping behaviour at a viscosity 1 GPas and 1 MPas, respectively.

Table IV.1 Vibration and damping parameters of material (b) for a wide range of viscosity values

Extensional viscosity	K	M	KM	Logarithmic decrement LD	Frequency (c/s)
1 GPas	$0.927.10^{-6}$	$0.762.10^{6}$	0.707	0.00162	544
1.05 MPas	0.841	0.841 (=K)	0.707	1.477	482
1 MPas	0.927	0.762	0.707	1.539	475
0.703 MPas	1.875	0.377	0.707	1.81 (max)	419
1 kPas	$0.927.10^{6}$	$0,762.10^{-6}$	0.707	0.00326	350

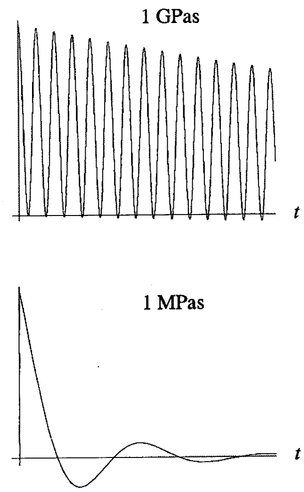

Figure IV.2 Free damped vibrations of material (b) at viscosity values 1 GPas and 1 MPas