Artificial orthotropy

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An homogeneous orthotropic plate model is presented for the idealisation of composed plate structures with orthotropy properties. The model requires the specification of 14 material parameters and couples membrane and bending forces. To find these parameters a finite element procedure is proposed that calculates these parameters with the help of a plane strain analysis and a simple potential analysis for a series of basic load cases. These parameters are substituted into the orthotropy model for the analysis of the homogeneous orthotropic plate. Both analysis steps can be performed with simple 2D finite element programs such as KOLA. Two examples show the performance of these models.

Key words: **orthotropic plates, finite element method, bending of plates, twisting of plates, composed structures, hollow core slabs**

1 Introduction

Many plate structures are built up in such a way that an isotropic idealisation of the properties is insufficient to model the structure properties properly. Quite often these structures, such as stiffened steel plates or concrete hollow core slabs or composed structures show an orthotropic behaviour of the properties.

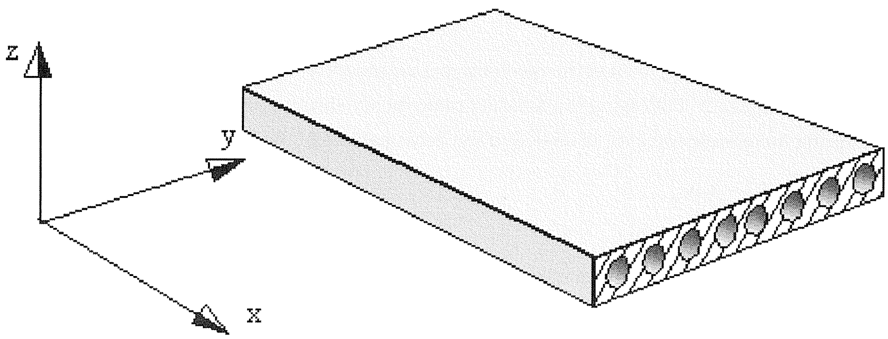

Figure 1

To model these structures by homogeneous orthotropic plates is an attractive solution. By such an approach very complicated slab structures can be analysed by simple 2D programs such as KOLA

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[4]. The problem, however, is the fixation of the material parameters. This paper presents a method, based upon the finite element method [1],[2], to calculate these properties in a numerical way.

2 The orthotropy properties

Following the (Reissner) plate bending theory we model the internal forces by membrane forces n_{xx}, n_{yy} and n_{xy}, plate moments m_{xx}, m_{yy} and m_{xy} and transverse shear forces q_x and q_y. These forces are dependent on the deformations of a reference plane of the slab, namely the strains ε^m_{xx}, ε^m_{yy} and γ^m_{xy}, the curvatures κ_{xx}, κ_{yy} and κ_{xy} and the shear deformations Ψ_x and Ψ_y. Assuming homogeneous orthotropy of the properties we have to formulate the constitutive equations.

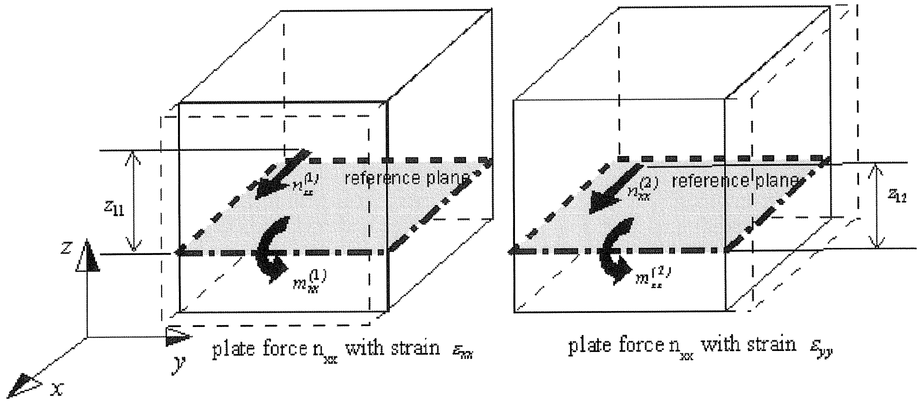


Figure 2

From constant strain ε_{xx} we get the resultant membrane force $n^{(1)}_{xx}$ at distance z_{11} from the reference plane -see Figure 2-. Following the Reissner plate theory strain ε_{xx} at the application point of the resulting membrane force $n^{(1)}_{xx}$ is related to the deformations of the reference plane by

$$\varepsilon_{xx} = \varepsilon^m_{xx} + z_{11}\kappa_{xx}$$

where $\kappa_{xx} = \phi_{y,x}$ and $\varepsilon^m_{xx} = u^m_{x,x}$

Now membrane force $n^{(1)}_{xx}$ is given by

$$n^{(1)}_{xx} = g_{11}\varepsilon^m_{xx} + z_{11}g_{11}\kappa_{xx} \qquad \text{(1a)}$$

Bending moment $m^{(1)}_{xx}$ at the reference plane is given by

$$m^{(1)}_{xx} = d_{11}\kappa_{xx} + z_{11}n^{(1)}_{xx}$$

or (1b)

$$m^{(1)}_{xx} = z_{11}g_{11}\varepsilon^m_{xx} + (d_{11} + z^2_{11}\, g_{11})\kappa_{xx}$$

From constant strain ε_{yy} we get a resulting membrane force $n^{(2)}_{xx}$ at a distance z_{12} from the reference plane. Unlike in the isotropic case usually $z_{12} \neq z_{11}$. Membrane force $n^{(2)}_{xx}$ is now given by

$$n^{(2)}_{xx} = g_{12}\varepsilon_{yy}$$

where (1c)

$$\varepsilon_{yy} = \varepsilon^{m}_{yy} + z_{12}\kappa_{yy}$$

The contribution to the bending moment at the reference plane is given by

$$m^{(2)}_{xx} = d_{12}\kappa_{yy} + z_{12}n^{(2)}_{xx}$$

or (1d)

$$m^{(2)}_{xx} = z_{12}g_{12}\varepsilon^{m}_{yy} + (d_{12} + z^{2}_{12}\, g_{12})\kappa_{yy}$$

In the same way we formulate the relations for membrane force n_{yy} and plate bending moment m_{yy}. Because of the reciprocity property it follows that $z_{21} = z_{12}$, $g_{21} = g_{12}$ and $d_{21} = d_{12}$.

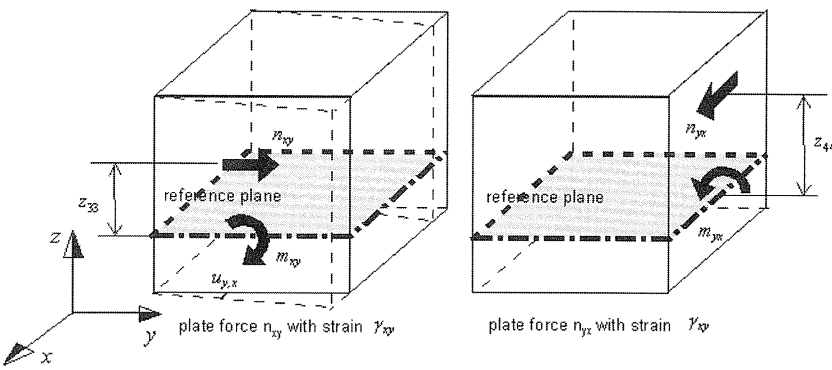


Figure 3

Shear forces n_{xy} and n_{yx} do not act at the same distance from the reference plane, thus $z_{33} \neq z_{44}$.
We find the displacements

$$u_x = u^{m}_x + z_{44}\phi_y$$

$$u_y = u^{m}_y - z_{33}\phi_x$$

Because rotation $\omega_z(z)$ is constant with respect to the thickness we may assume that $\phi_{x,x} = -\phi_{y,y}$
Because of equilibrium we may assume that $n_{xy} = n_{yx}$.
Now we can derive

$$n_{xy} = g_{33}(u^{m}_{x,y} + u^{m}_{y,x}) - z_{33}g_{33}\phi_{x,x} + z_{44}g_{33}\phi_{y,y}$$

$$n_{yx} = g_{33}(u^{m}_{x,y} + u^{m}_{y,x}) - z_{33}g_{33}\phi_{x,x} + z_{44}g_{33}\phi_{y,y}$$

$$m^m_{xy} = z_{33} g_{33} u^m_{x,y} + z_{33} g_{33} u^m_{y,x} - (d_{33} + z^2_{33} g_{33})\phi_{x,x} + z_{33} z_{44} g_{33} \phi_{y,y}$$
$$m^m_{yx} = z_{44} g_{33} u^m_{x,y} + z_{44} g_{33} u^m_{y,x} - z_{33} z_{44} g_{33} \phi_{x,x} + (d_{44} + z^2_{44} g_{33})\phi_{y,y}$$

After some manipulations we get

$$n_{xy} = g_{33}\gamma^m_{xy} + \tfrac{1}{2}(z_{33} + z_{44}) g_{33} \kappa_{xy}$$
$$\tfrac{1}{2}(m_{xy} + m_{yx}) = \tfrac{1}{2}(z_{33} + z_{44}) g_{33} \gamma^m_{xy} + \{\tfrac{1}{2}(d_{33} + d_{44}) + \tfrac{1}{4}(z_{33} + z_{44})^2 \, g_{33}\}\kappa_{xy} \tag{2}$$
$$\text{where } \kappa_{xy} = -\phi_{x,x} + \phi_{y,y} \text{ and } \gamma^m_{xy} = u^m_{x,y} + u^m_{y,x}$$

It has to be noted that, although z_{33} and z_{44} are dependent on y, the sum $z_{33} + z_{44}$ is independent of y. This holds also for d_{33} and d_{44}.
In accordance to the Reissner plate theory we find for the transverse shear forces q_x and q_y

$$q_x = g_{44}\psi_x$$
$$q_y = g_{55}\psi_y \tag{3}$$
$$\text{where } \psi_x = u^m_{z,x} + \phi_y \text{ and } \psi_y = u^m_{z,y} - \phi_x$$

Summarising all contributions together we find the constitutive equations

$$\begin{bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \\ m^m_{xx} \\ m^m_{yy} \\ \hat{m}^m_{xy} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & 0 & z_{11}g_{11} & z_{12}g_{12} & 0 \\ g_{12} & g_{22} & 0 & z_{12}g_{12} & z_{22}g_{22} & 0 \\ 0 & 0 & g_{33} & 0 & 0 & \hat{z}_{33}g_{33} \\ z_{11}g_{11} & z_{12}g_{12} & 0 & d_{11} + z^2_{11}g_{11} & d_{12} + z^2_{12}g_{12} & 0 \\ z_{12}g_{12} & z_{22}g_{22} & 0 & d_{12} + z^2_{12}g_{12} & d_{22} + z^2_{22}g_{22} & 0 \\ 0 & 0 & \hat{z}_{33}g_{33} & 0 & 0 & \hat{d}_{33} + \hat{z}^2_{33}g_{33} \end{bmatrix} \begin{bmatrix} \varepsilon^m_{xx} \\ \varepsilon^m_{yy} \\ \gamma^m_{xy} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$
$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} g_{44} & 0 \\ 0 & g_{55} \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} \tag{4}$$

where

$$\hat{m}^m_{xy} = \tfrac{1}{2}(m^m_{xy} + m^m_{yx})$$
$$\hat{z}_{33} = \tfrac{1}{2}(z_{33} + z_{44})$$
$$\hat{d}_{33} = \tfrac{1}{2}(d_{33} + d_{44})$$

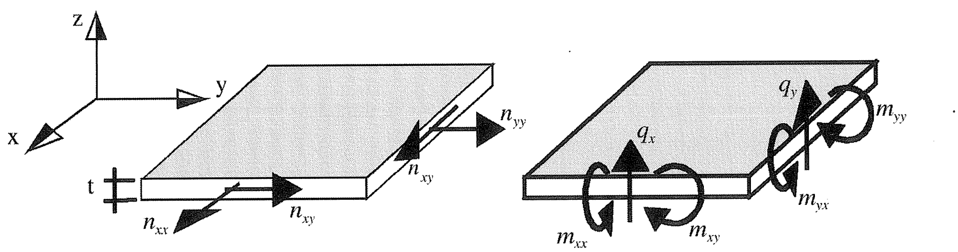


Figure 4. Membrance forces, plate moments and plate transverse shear forces.

Kinematic relations are given by

$$
\begin{aligned}
\varepsilon^m_{xx} &= u^m_{x,x} \\
\varepsilon^m_{yy} &= u^m_{y,y} \\
\gamma^m_{xx} &= u^m_{x,y} + u^m_{y,x} \\
\kappa_{xx} &= \phi_{y,x} \\
\kappa_{yy} &= -\phi_{x,y} \\
\kappa_{xy} &= -\phi_{x,x} + \phi_{y,y} \\
\psi_x &= u_{z,x} + \phi_y \\
\psi_y &= u_{z,y} - \phi_x
\end{aligned}
\qquad (5)
$$

It is the objective of this study to develop a method for the fixation of the parameters g_{ij}, d_{ij} and z_{ij}.

3 The finite element models

To smooth the properties of an inhomogeneous orthotropic plate into a continuous orthotropic Reissner model we will perform a series of calculations with the help of the finite element method. In this way we will calculate the 14 parameters g_{ij}, d_{ij} and z_{ij}.

For our analyses we will define a series of loading cases in such a way that all strains are either constant with respect to the axial direction x or equal to zero. For every case it will hold

$$\frac{\partial}{\partial x}(u_{i,j}) = 0 \qquad (6)$$

The equilibrium of stresses in the x-direction (axial axis) requires

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + p_x = 0$$

which yields, after substitution of isotropic stress strain relations and condition (6), a potential equation in displacement u_x [3],[5]

$$Gu_{x,yy} + Gu_{x,zz} + p_x = 0 \qquad (7)$$

The equilibrium of stresses in y- and z-directions requires

$$
\begin{aligned}
\sigma_{yy,y} + \sigma_{yz,z} + p_y &= 0 \\
\sigma_{yz,y} + \sigma_{zz,z} + p_z &= 0
\end{aligned}
\qquad (8)
$$

Assuming a plane strain model in the yz-plane, thus $\varepsilon_{xx} = 0$ we follow the constitutive equations

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\{(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}\}$$
$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\{\nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}\} \tag{9}$$
$$\sigma_{yz} = G\gamma_{yz}$$

Following these conditions the displacement $u_x(y, z)$ is found completely independent from the displacements $u_y(y, z)$ and $u_z(y, z)$. We will define a series of load cases and boundary conditions that satisfy to these conditions. The problems that are formulated either by the plane strain model or the potential problem, are solved with the help of the finite element method. In all cases we will assume a constant plate thickness t=1.

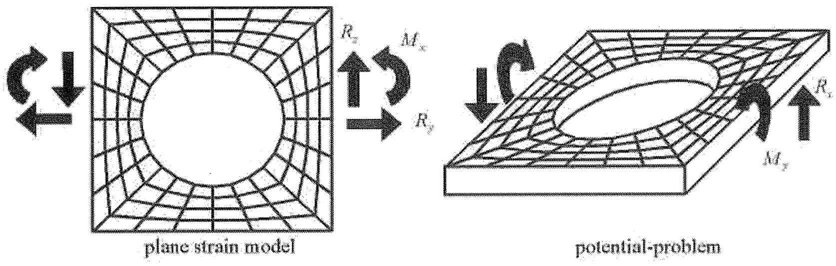


Figure 5. Finite element models for one segment of the cross section

The finite element analysis will yield reaction forces at the left hand side and the right hand side of a repeating cross section segment, e.g. as shown in figure 5. From these reaction forces the resultants at both sides are calculated following

$$\left.\begin{aligned} n_{yy} &= R_y = \sum_i F_{y_i} \\ q_y &= R_z = \sum_i F_{z_i} \\ m_{yy} &= -M_x = \sum_i z_i F_{y_i} \end{aligned}\right\} \text{ plane strain model}$$
$$\left.\begin{aligned} n_{xy} &= R_x = \sum_i F_{x_i} \\ m_{xy} &= M_y = \sum_i z_i F_{x_i} \end{aligned}\right\} \text{ potential problem} \tag{10}$$

The calculation of the plate forces n_{xx}, q_x and m_{xx} will be discussed later.

4 Analyses with the plane strain model

Load case 1: Simulation ε_{yy}, calculation g_{22} and z_{22}

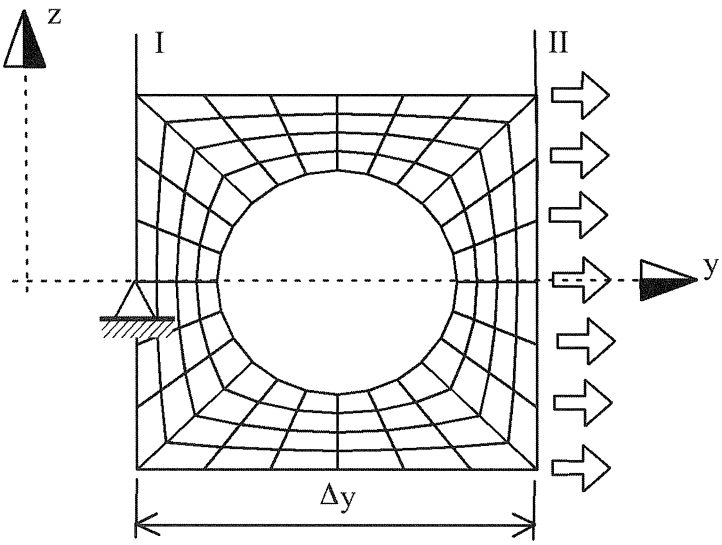


Figure 6

To simulate strain ε^{m}_{yy} we apply the kinematic constraints

$$
\begin{aligned}
u_y^{II} &= u_y^{I} + \Delta u_y \\
u_z^{II} &= u_z^{I}
\end{aligned}
\tag{11}
$$

The use of these kinematic constraints does not hinder the development of undisturbed warping of the edges I and II, thus it is possible that $u_y^{I} \neq 0$ and $u_z^{I} \neq 0$.
For the homogeneous plate we substitute

$$
\varepsilon^{m}_{yy} = \frac{\Delta u_y}{\Delta y}
$$

$$
\varepsilon^{m}_{zz} = 0
$$

Now we get resulting force R_y which equals to $R_y = n_{yy} = g_{22}\varepsilon^{m}_{yy} = g_{22}\dfrac{\Delta u_y}{\Delta y}$ .and $M_x = -z_{22}R_y$
Using these results we calculate the parameters

$$
\begin{aligned}
g_{22} &= \frac{\Delta y \, R_y}{\Delta u_y} \\
z_{22} &= -\frac{M_x}{R_y}
\end{aligned}
\tag{12}
$$

Load case 2: Simulation κ_{yy}, calculation d_{12}

We will simulate the pure bending case. The load case is an increment $\Delta\phi_x$ of the rotation between edge I and II. To simulate the pure bending κ_{yy} we apply the following kinematic constraints:

$$u^{II}_{y_i} = u^{I}_{y_i} - (z_i - z_{22})\Delta\phi_x \qquad (13)$$
$$u^{II}_{z_i} = u^{I}_{z_i}$$

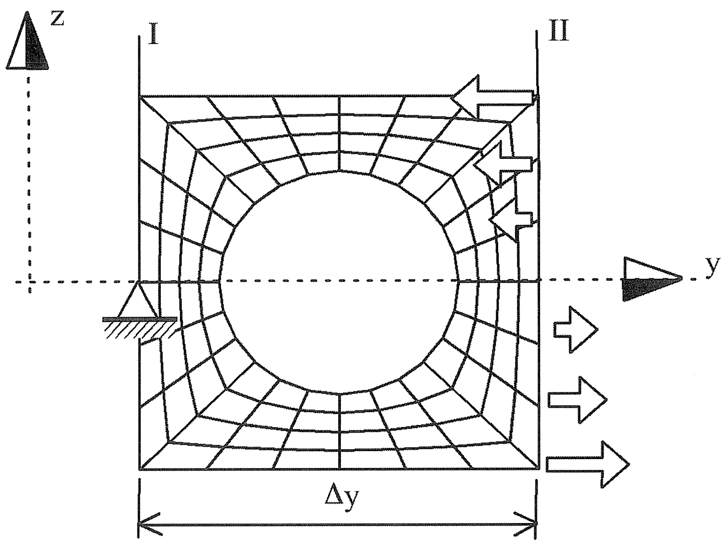


Figure 7

For the homogeneous plate we assume

$$\kappa_{yy} = -\phi_{x,y} = -\frac{\Delta\phi_x}{\Delta y} \qquad (14)$$

Given these boundary conditions the plate moment m_{yy} at the right hand edge equals to:

$$m^{II}_{yy} = -d_{22}\frac{\Delta\phi_x}{\Delta y} \qquad (15)$$

From the results of our f.e.m. analysis we compute reaction force M_x. Because $M_x = -m_{yy}$ we find

$$d_{22} = \frac{\Delta y\, M_x}{\Delta\phi_x} \qquad (16)$$

Load case 3: Simulation ψ_y, calculation g_{55}

To calculate the shear stiffness g_{55} we look for a loading case that represents shear deformation only. Such a case is possible by the simulation of a Timoshenko beam which is subjected to an uniformly distributed moment load m_x. The Timoshenko beam has to satisfy the differential equation

$$q_y = m_{yy,y} + m_x$$

Now we apply the boundary conditions

$$u_z^I = u_z^{II} = 0$$

$$\phi_x^I = \phi_x^{II}$$

The solution of this differential equation and boundary conditions is given by

$$m_{yy} = 0 \text{ and } q_y = m_x .$$

The deformation is fixed by $q_y = g_{55}\Psi_y$

To simulate the Timoshenko beam by a f.e.m. analysis we take the load case

$$p_y(y,z) = -\sigma_{yy}(y,z) \tag{17}$$

where σ_{yy} is found by load case 2.

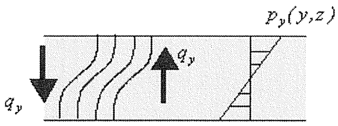


Figure 8

The resultant loading of (17) at the y-axis is the uniformly distributed moment

$$m_x(y) = \int z\sigma_{yy} dz = M_x = \text{constant} \tag{18}$$

The boundary conditions are given by kinematic constraints

$$u_{v_i}^{II} = u_{v_i}^{I}$$

$$u_{z_i}^{II} = u_{z_i}^{I}$$

The homogeneous plate model takes into account only a constant shear force. Based upon the virtual work condition we require that

$$\Delta y \; q_y \psi_y = \sum_e \boldsymbol{u}^{e^T} \boldsymbol{K}^e \boldsymbol{u}^e = \sum_e \boldsymbol{u}^{e^T} \boldsymbol{f}^e$$

or

$$\Delta y \frac{q_y^2}{g_{55}} = \boldsymbol{u}^T \boldsymbol{f}$$

Now we find the parameter g_{55} by

$$g_{55} = \frac{\Delta y\, q_y^2}{\mathbf{u}^T \mathbf{f}} \tag{19}$$

Load case 4: Simulation ε_{xx}, calculation g_{11}, g_{12}, z_{11} and z_{12}

For materials with Poissons ratio $\nu = 0$ the analysis is very simple. We find

$g_{11}\Delta y = EA$ and $g_{12} = 0$.

thus

$$g_{11} = \frac{EA}{\Delta y}$$

$$g_{12} = 0\,.$$

The calculation of eccentricities z_{11} and z_{11} can be more complicated and justifies the following procedure.

The problem is more complicated with $\nu \neq 0$. In those cases neither the onedimensional stress assumption nor the plane stress assumption is correct. For a correct analysis the strains ε_{yy}, ε_{zz} and γ_{yz} are calculated with a f.e.m. analysis of the plane strain model with initial strain loads.
We apply the following initial strains.

$$\varepsilon_{xx}^0 = \Delta u_x \quad \Delta u_y = 0 \quad \Delta u_z = 0 \quad \Delta\phi_x = 0 \quad \Delta\phi_y = 0 \tag{20}$$

This corresponds with the following initial stresses

$$\begin{aligned}
\sigma_{xx}^0 &= \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\Delta u_x \\
\sigma_{yy}^0 &= \frac{\nu E}{(1+\nu)(1-2\nu)}\Delta u_x \\
\sigma_{zz}^0 &= \frac{\nu E}{(1+\nu)(1-2\nu)}\Delta u_x
\end{aligned} \tag{21}$$

After performance of this analysis we get at boundary II the resultant forces $n_{yy} = R_y$ and $m_{yy} = -M_x$ where

$$n_{yy} = g_{12}\Delta u_x$$

$$m_{yy} = z_{12} n_{yy}$$

In this way we find the parameters d_{12} and z_{12} following

$$\begin{aligned}
g_{12} &= \frac{R_y}{\Delta u_x} \\
z_{12} &= -\frac{M_x}{R_y}
\end{aligned} \tag{22}$$

For the plane strain model that we analysed we find for stresses σ_{xx}

$$\sigma_{xx} = \sigma^{0}_{xx} + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{yy} + \varepsilon_{zz})$$

From these stresses we get n_{xx} following:

$$n_{xx}\Delta y = \sum_{e}\iint_{A^e} \sigma^{e}_{xx}\, dA = g_{11}\Delta y \;\; \Delta u_{x} \tag{23}$$

and plate moment m_{xx} following

$$\Delta y\, m_{xx} = \sum_{e}\iint z\sigma^{e}_{xx}\, dA = z_{11} n_{xx}\Delta y \tag{24}$$

Thus parameters g_{11} and z_{11} are found by

$$g_{11} = \frac{n_{xx}}{\Delta u_{x}}$$

$$z_{11} = \frac{m_{xx}}{n_{xx}}$$

Load case 5: Simulation κ_{xx}, calculation d_{11} and d_{12}

For simple structures with $\nu = 0$ it is simple to calculate d_{11} and d_{12}. For those cases is

$$d_{11}\Delta y = EI, \text{ thus } d_{11} = \frac{EI}{\Delta y}, \text{and } d_{12} = 0\,.$$

For more complicated cross sections the underlying procedure is recommended.

To simulate a pure bending in axial direction we apply the following constraints

$$\varepsilon^{0}_{xx} = (z - z_{11})\Delta\phi_{y} \quad \Delta u_{y} = 0 \quad \Delta u_{z} = 0 \quad \Delta\phi_{x} = 0 \tag{25}$$

This corresponds with the initial stresses

$$\begin{aligned}
\sigma^{0}_{xx} &= \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}(z - z_{11})\Delta\phi_{y} \\
\sigma^{0}_{yy} &= \frac{\nu E}{(1+\nu)(1-2\nu)}(z - z_{11})\Delta\phi_{y} \\
\sigma^{0}_{zz} &= \frac{\nu E}{(1+\nu)(1-2\nu)}(z - z_{11})\Delta\phi_{y}
\end{aligned} \tag{26}$$

After solution of this plane strain problem we get $m_{yy} = -M_{x}$ at boundary II where

$$m_{yy} = d_{12}\Delta\phi_{y}$$

We find parameter d_{12} following

$$d_{12} = -\frac{M_x}{\Delta\phi_y} \qquad (27)$$

After solution of the plane strain problem we get for the axial stresses σ_{xx}

$$\sigma_{xx} = \sigma^0_{xx} + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{yy} + \varepsilon_{zz}) \qquad (28)$$

From these stresses we get a resulting moment M_y following:

$$M_y = \Delta y\ m_{xx} = \sum_e \iint_{A^e} (z - z_{11})\sigma^e_{xx}\, dA = d_{11} \Delta y\ \Delta\phi_y \qquad (29)$$

The parameter d_{11} is now given by

$$d_{11} = \frac{m_{xx}}{\Delta\phi_y}$$

5 Analyses with the potential problem

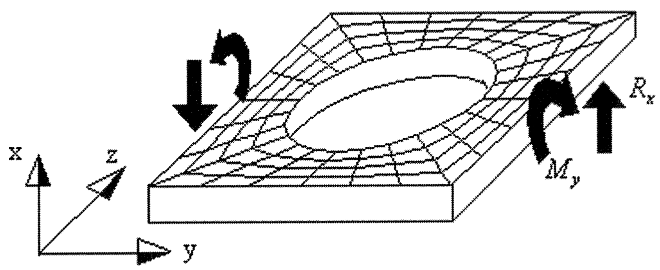


Figure 9

Load case 6: Simulation Ψ_x, calculation g_{44}

For the calculation of shear stiffness g_{44} we look for a loading case that generates shear deformation only. Similar to load case 3 we apply an uniformly distributed moment load m_y following

$$m_y = \iint_{A^e} \sigma_{xx}(y,z)\, z\, dA$$

where $\sigma_{xx}(y,z)$ is found in (28) from an analysis with constant bending moment m_{xx}. In the f.e.m. model we apply the load $p_x(y,z) = \sigma_{xx}(y,z)$ such as found in (28)

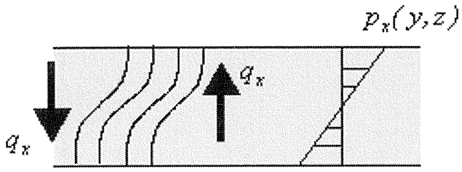


Figure 10

For this load case we find a solution q_x such that

$$q_x \Delta y = m_y = \iint_{A^e} \sigma_{xx}(y,z)\, z\, dA \tag{30}$$

Because any variation with respect to x equals to zero the equilibrium transforms into the potential problem -see also (7)-

$$Gu_{x,yy} + Gu_{x,zz} + p_x = 0 \tag{31}$$

For this potential problem we apply the kinematic boundary conditions

$$u_x^I = u_x^{II}$$

After solution of the potential problem we substitute $\sigma_{xx} = Gu_{x,z}$ and $\sigma_{xy} = Gu_{x,y}$ and successively

$$\Delta y\ q_x \psi_x = \sum_e \boldsymbol{u}^{e^T} \boldsymbol{K}^e \boldsymbol{u}^e = \sum_e \boldsymbol{u}^{e^T} \boldsymbol{f}^e \tag{32}$$

or

$$\Delta y \frac{q_x^2}{g_{44}} = \boldsymbol{u}^T \boldsymbol{f}$$

Now we find the parameter g_{44} by

$$g_{44} = \frac{\Delta y\ q_x^2}{\boldsymbol{u}^T \boldsymbol{f}} \tag{33}$$

With the results for σ_{xz} we can calculate the y-ordinate y_{33} of the shear force centre following

$$y_{33} = \frac{\sum_e \iint y \sigma_{xz}^e dA}{q_x \Delta y}$$

Load case 7: Simulation γ_{xy}, calculation g_{33} and z_{33}

In the potential problem we apply the kinematic constraints

$$u_x^{II} = u_x^I + \Delta u_x \tag{34}$$

For the homogeneous plate we simulate

$$\gamma^{m}_{xy} = \frac{\Delta u_x}{\Delta y} \tag{35}$$

After solution of the potential problem we find the resulting force

$$R_x = n_{yx} = g_{33}\gamma^{m}_{xy} = g_{33}\frac{\Delta u_x}{\Delta y}\ .$$

and

$$M_y = z_{44} R_x$$

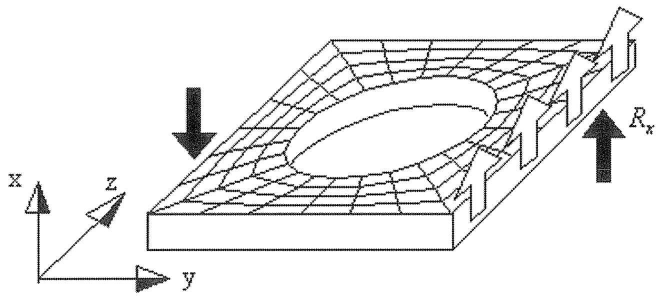


Figure 11

From these results we obtain the parameters

$$g_{33} = \frac{\Delta y\ R_x}{\Delta u_x}$$

$$z_{44} = \frac{M_y}{R_x} \tag{36}$$

The second part of the postprocess requires

$$M_x = \sum_e \iint (\,y\sigma_{xz} - z\sigma_{xy}\,)dA$$

and yields the parameter

$$z_{33} = \frac{M_x}{R_x \Delta y}$$

Load case 8: Simulation κ_{xy}, calculation d_{33}

To calculate twisting stiffness d_{33} we investigate a plate with constant twisting moment m_{xy}. Because the plate is free of shear forces it holds (5) that $\phi_x = u_{z,y}$ and $\phi_y = -u_{z,x}$. With undisturbed warping there is no interaction between the shear stresses σ_{xz} and σ_{xy} the cross section stresses σ_{yy}, σ_{zz} and σ_{yz}. The cross section rotates and translates as a rigid body.

As loading case we apply a distorsion $\phi_{x,x} = \Delta\phi_x$ around the shear force centre of a segment of the cross section. Taking the origin of the yz-reference frame in the shear force centre we get the displacements

$$
\begin{aligned}
u_y &= -xz\Delta\phi_x \\
u_z &= xy\Delta\phi_x \\
\Delta\phi_y &= -\Delta y\ \Delta\phi_x
\end{aligned}
\tag{37}
$$

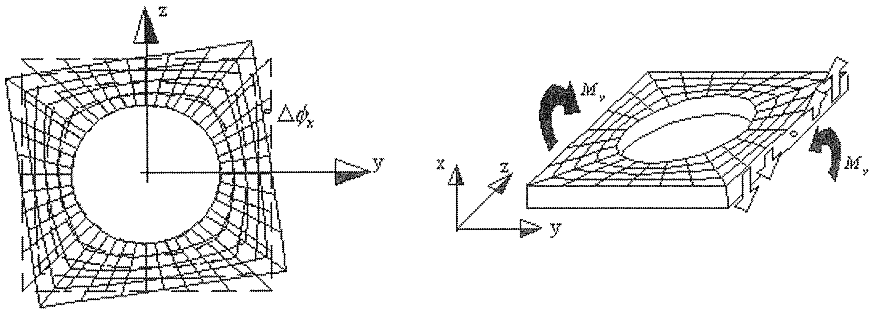


Figure 12

To these displacements we find the shear stresses

$$\sigma_{xz} = Gu_{x,z} + Gy\Delta\phi_x$$

$$\sigma_{xy} = Gu_{x,y} - Gz\Delta\phi_x$$

For the potential problem [3], [5] we have to satisfy the boundary condition

$$\sigma_{xn} = Gu_{x,n} + Gy_n\Delta\phi_x = 0 \tag{38}$$

This boundary condition can be translated as a 'load' at the boundaries of the potential problem.

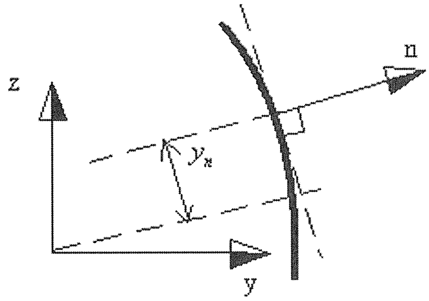


Figure 13

Together with this 'load' we have to satisfy to the kinematic constraints

$$u_x^{II} = u_x^{I} + z\Delta y\ \Delta\phi_x$$

After solution of the potential problem we calculate the twisting moment following

$$\Delta y\ m_{xy} = \sum_e \iint (\bar{\sigma}_{xy} z - \bar{\sigma}_{xz} y)\, dA = d_{33}\kappa_{xy} = 2\Delta y\ d_{33}\Delta\phi_x$$

which yields the last parameter

$$d_{33} = \frac{m_{xy}}{2\Delta\phi_x} \qquad (39)$$

6 Examples

Calculation properties

To test the theory such as outlined in the preceding sections we will analyse a stiffened structure as shown in figure 14.

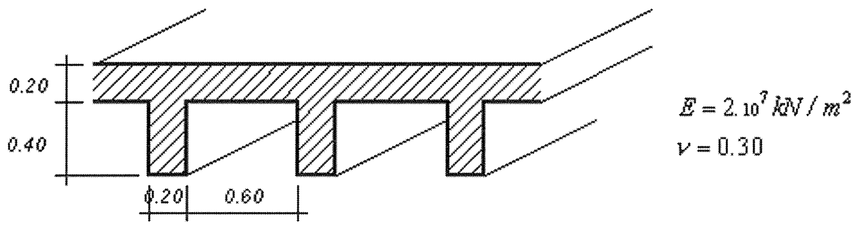


Figure 14

The cross section by 14 segments as shown in figure 15

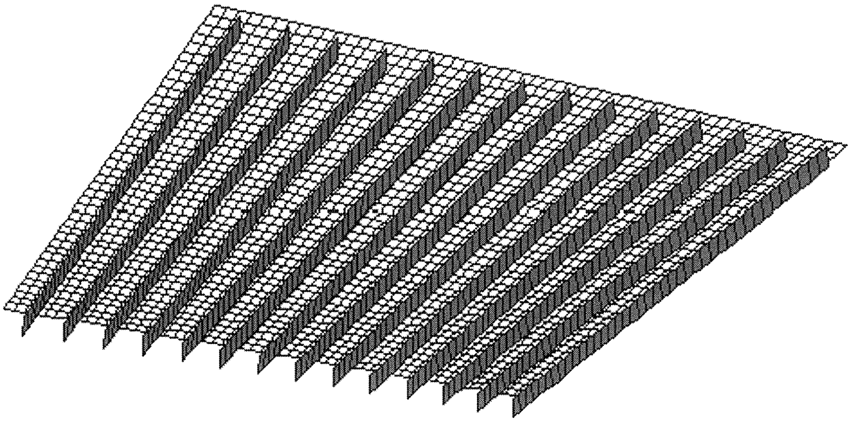


Figure 15. Model of isotropic slab stiffened with beams

The structure will be modelled in three ways, namely

- By an isotropic slab stiffened by beams -see figure 15-
- By an artificial orthotropic plate
- By a homogeneous isotropic slab with thickness 0.30 m

To apply artificial orthotropy we model one segment with the finite element method. The mesh is shown in figure 16.

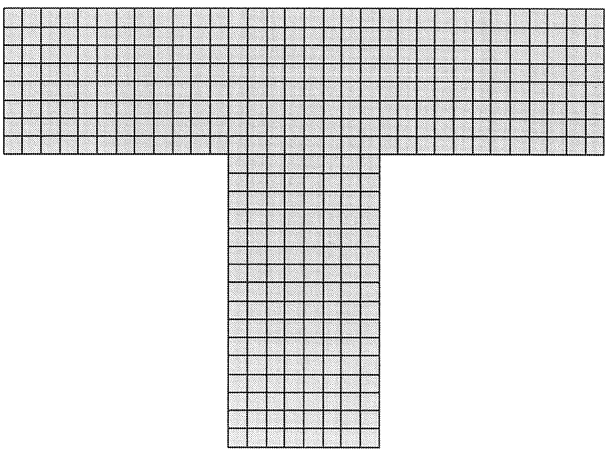


Figure 16. One segment of the cross section.

After solution of the analysis with the 8 load cases we get the following properties:

kN/m	kN	m
$g_{11} = 0.64029\ 10^{7}$	$d_{11} = 0.16502\ 10^{6}$	$z_{11} = 0.40618$
$g_{12} = 0.13431\ 10^{7}$	$d_{12} = 0.46507\ 10^{4}$	$z_{12} = 0.49802$
$g_{22} = 0.44771\ 10^{7}$	$d_{22} = 0.15763\ 10^{5}$	$z_{22} = 0.49802$
$g_{33} = 0.16026\ 10^{7}$	$d_{33} = 0.11652\ 10^{5}$	$z_{33} = 0.48737$
$g_{44} = 0.10080\ 10^{7}$	$d_{44} = 052779\ 10^{4}$	$z_{44} = 0.49990$
$g_{55} = 0.13437\ 10^{7}$		

These properties are input for the analysis with artificial orthotropic plates

Test on axial stiffness

A 20 m long and 11.20 m wide slab is supported at the short edges. The bridge is loaded by a point load P=10 kN at the centre of the slab.

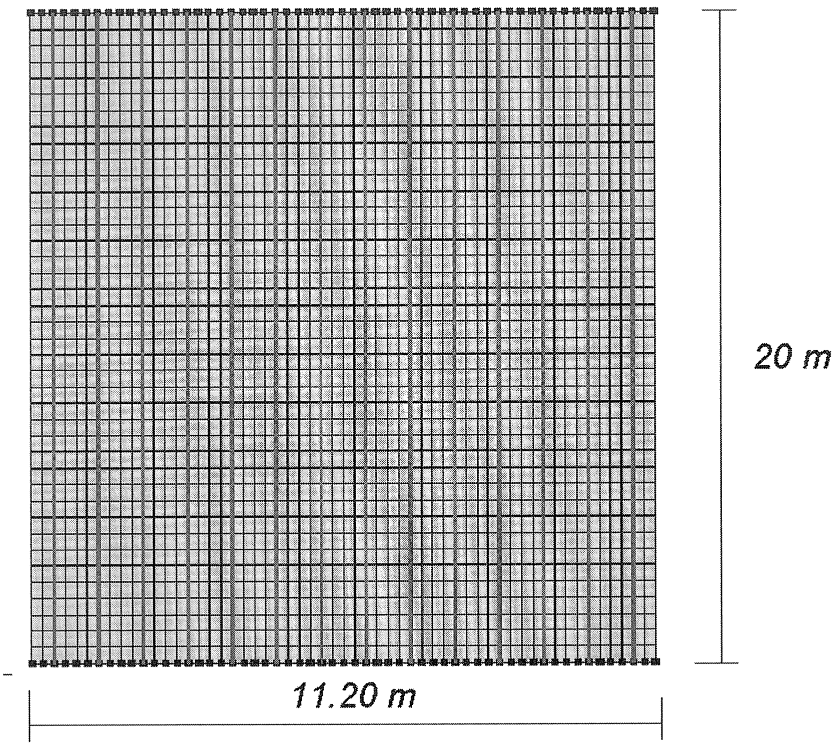


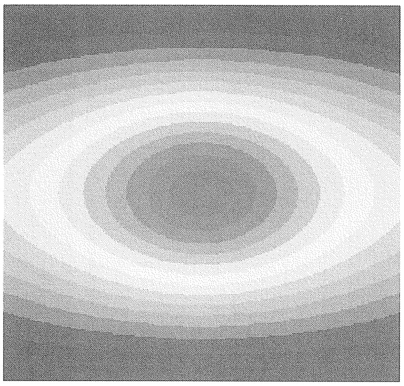
Figure 17

The analyses carried out with the three models yield the following results for the displacements under the load

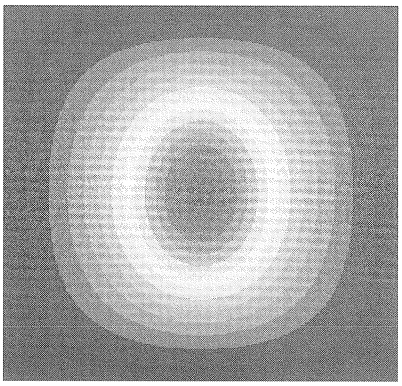
- Isotropic plate stiffened by beams $u_z = 0{,}171\ mm$

- Artificial orthotropic plate u_z = 0,170 *mm*
- Homogeneous isotropic plate with thickness 0.30 m u_z = 0,251 *mm*

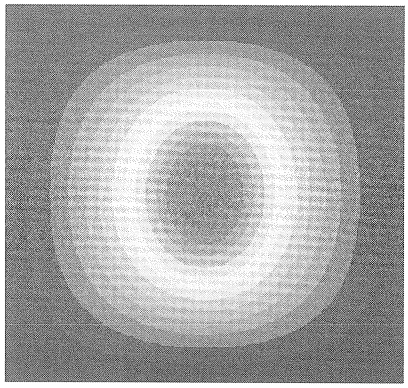
Global results are shown in figure 18.



homogeneous isotropic plate



isotropic plate with beams



artificial orthotropic plate

Figure 18. Displacements u_z

The reaction forces for each model are shown in figure 19

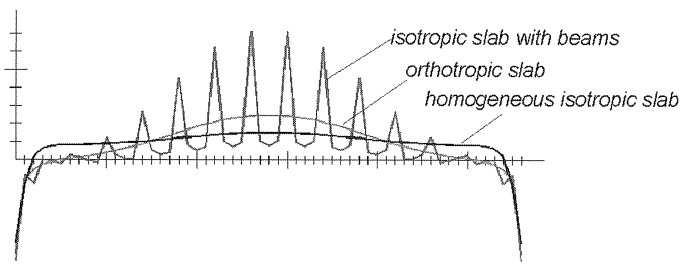


Figure 19. Reaction forces.

Test on twisting stiffness

To test the twisting moment stiffness we take a quarter of the slab and support the slab along one axial and one cross direction border -see figure 20-

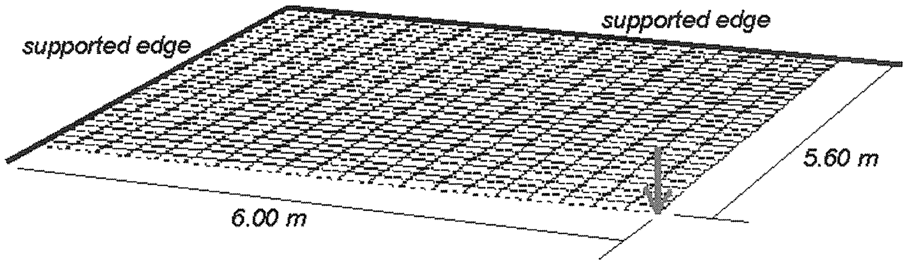


Figure 20

The analyses carried out with the three models yield the following results for the displacements under the load

- Isotropic plate stiffened by beams $u_z = 3.98\ mm$
- Artificial orthotropic plate $u_z = 3.46\ mm$
- Homogeneous isotropic plate with thickness 0.30 m $u_z = 2.26\ mm$

The difference between the orthotropic slab and the slab with beams is rather large. An explanation is found by a consideration of the shear strain energy with pure twisting (load case 8).

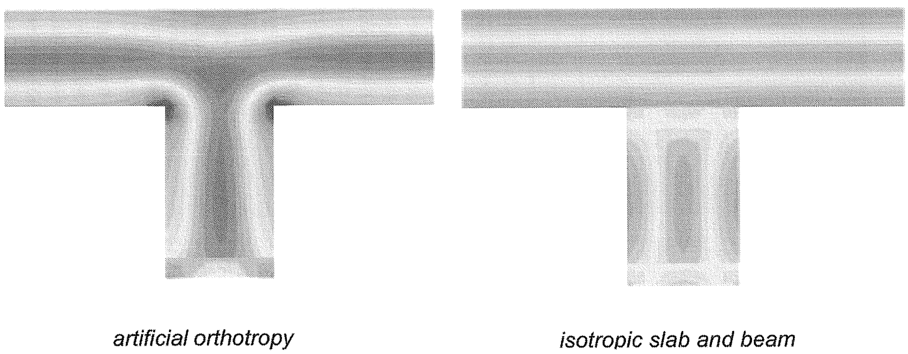


Figure 21. Maximum shear stresses in cross section.

Following the plate model stiffened by beams the transverse shear stresses of beam and slab for twisting are not coupled at the junction of beam and slab; the model does not take into account this condition -see figure 21-. Both models are subjected to the same loads. A simple estimate of the strain energy shows a reduction of 12% of the orthotropic model with respect to the stiffened slab model. Thus also the displacements of the stiffened slab model have to be reduced with 12%. Now we find the improved estimate

- Isotropic plate stiffened by beams $u_z = 350$ mm

which is much closer to the orthotropic model.

Global results are shown in figure 22

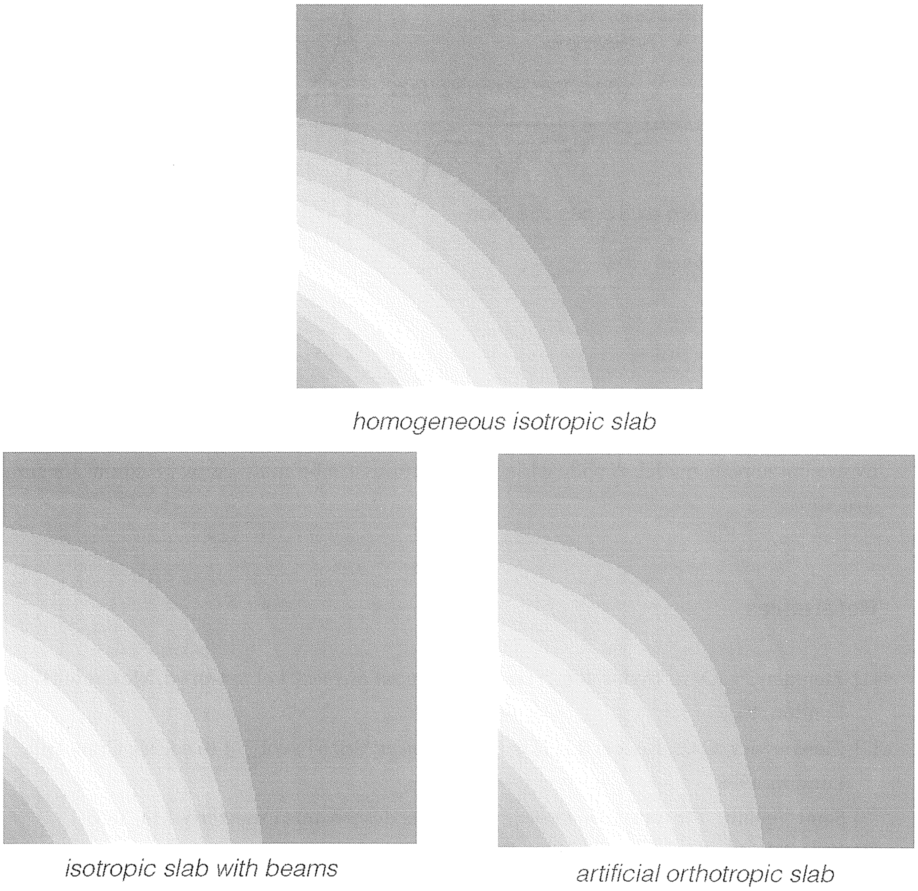


Figure 22. Displacements u_z

The reaction forces are given in figure 23

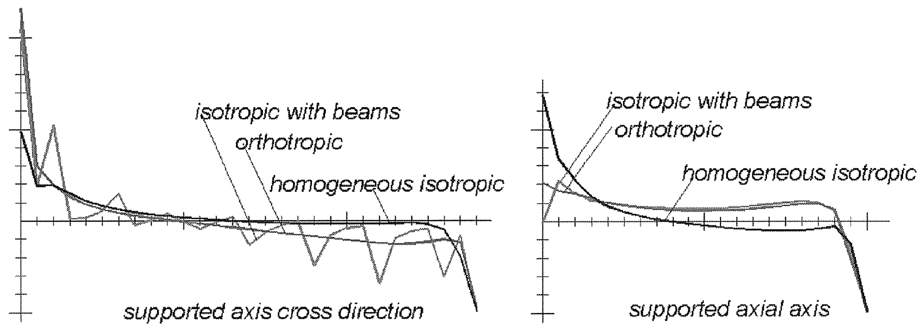


Figure 23. Reaction forces

7 Conclusions

The introduction of artificial orthotropy offers the tools to model complex slab structures by a relatively simple plate model. A wide variety of structures can be analysed by programs for simple 2D structures.

References

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Notations and symbols

Δy length segment
t thickness plate
E Young's modulus
G shear modulus
V Poisson's ratio
u displacement
ϕ rotation
ε strain

κ curvature

ψ shear deformation

Δu elongation

$\Delta\phi$ distorsion

σ stress

n membrane force

m plate bending moment

q plate transverse shear force

R resultant force at boundary

M resultant moment at boundary

s_{ij} stiffness stresses

d_{ij} stiffness plate bending

g_{ij} stiffness plate forces

$\boldsymbol{u}$ displacement vector

$\boldsymbol{f}$ force vector

$\boldsymbol{K}$ stiffness matrix

$\boldsymbol{u}^T\boldsymbol{f}$ vector product $\boldsymbol{u}$ and $\boldsymbol{f}$

$u_{,x}$ differentiation of u with respect to x