A Review of Methods for Constrained Eigenvalue Problems

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**Numerous studies are concerned with vibrations or buckling of constrained systems realising that complex systems can be analysed when starting from unconstrained or known problems.
Over the years various constrained eigenvalue formulations have been published and put into practical use. The most important ones are the Lagrangian multiplier method (modal synthesis method or component modes synthesis method), the receptance method and the modal constraint method. In this paper the similarities and merits of the various methods are discussed.
It is striking that so far the similarities of the various constrained eigenvalue expressions have not been reported nor has the similarity of the eigenvalue expressions of these methods with Weinstein's determinant for intermediate problems of the first type been noticed.
The eigenvalue formulations of the Lagrangian multiplier method and that of the receptance appear to be similar to Weinstein's determinant for intermediate problems of the first type. The modal constraint method is based on an extension of Weinstein's method for intermediate problems of the first type and offers some significant advantages, i.e. the resulting eigenvalue formulation of the modal constraint method has a standard form in contrast with that of the other mentioned methods in that they have the known and unknown eigenvalues in the denominator of the eigenvalue formulations. Further, zero modal displacements, persistent and multiple eigenvalues do require special attention using these methods whereas this is not the case for the modal constraint method.
Based on the similarities of the various constrained eigenvalue expressions a number of interesting conclusions are drawn.**

Keywords: vibration, buckling, constraints, component mode synthesis, modal synthesis method, receptance method, modal constraint method.

1 Introduction

There are a number of methods that deal with the problem of constrained vibrations, i.e. the vibrations of constrained linear mechanical systems.
The eigenvalue equations of these methods, notably the receptance method [5] and the Lagrangian multiplier method [6] do not allow persistent eigenvalues readily to be calculated, i.e. eigenvalues common to both the unconstrained and the constrained problem to be computed. This rule and the resulting eigenvalue equations of these methods do not allow the

use of standard numerical eigenvalue solvers. Here root finding procedures like the bi-section methods have to be applied. However persistent and multiple eigenvalues cannot be calculated using these methods. Some possible approaches to tackle these problems will be discussed. Furthermore, the eigenfunctions, i.e. the Lagrangian multipliers or receptances of these eigenvalue formulations do not directly allow constrained mode shapes to be evaluated. Notwithstanding these observations, the Lagrangian method [11-17] and the receptance method [18-23] are widely used without realising that both methods appear to be in essence similar. The modal constraint method has been developed starting from Weinstein's determinant of the first type and does not suffer from the abovementioned drawbacks [7-10].

There is a need to discuss the similarities, differences, merits and demerits of the various methods for constrained eigenvalue problems.

In section 2 the basics of the Lagrangian method will be presented resulting in an constrained eigenvalue expression. The drawbacks of this method will be summarised. Also the boundedness of eigenvalues is discussed.

Thereafter in section 3 the receptance method will be discussed. The constrained eigenvalue expression is similar to the one for the Lagrangian multiplier method altough both methods depart from entirely different approaches.

Then, in section 4 Weinstein's method of intermediate problems of the first type will be treated. It will be shown that the Lagrangian multiplier method and the receptance method that deal with constrained vibrating problems are in fact similar to Weinstein's method of the first type, i.e. their eigenvalue formulations have similar formulations. Weinstein's method is firmly based on the principles of differential operators and presents a stepping stone in the proof of the observed boundedness of constrained eigenvalues.

Weinstein's theory is based on arbitrary constraint functions. In section 5 a particular choice for these constraint functions will be given that relates to physical constraint conditions such as point supports.

Thereafter, in section 6 the modal constraint method based on a differential operator basis will be presented which allow persistent eigenvalues and also mode shapes to be established. Here it will be demonstrated how Lagrangian multipliers can be expanded into known quantities, which is the essence of the modal constraint method.

In section 7 the modal constraint method is treated in terms of energy functionals. It appears that the Lagrangian multipliers can be identified as the inner product of the constrained operator and the constraint functions.

In section 8 the similarities, differences, merits and demerits of constrained eigenvalue problems are summarised.

2 The Lagrangian Multiplier Method

2.1 ***Introduction***

The treatment of the Lagrangian multiplier method by Dowell [6] has been inspired by the early work of Budianski and Hu [4]. The latter author treat buckling problems for constrained structures using Lagrangian multipliers. He also realised that this method can also be applied to free vibration problems.

The free vibrations of an arbitrary structure in terms of component modes are determined by the use of normal modes of an unconstrained structure.

This is achieved by enforcing continuity conditions using Lagrangian multipliers. Dowell regards the method as a constrained Rayleigh-Ritz method with constraint or continuity conditions.

It is well known that the Rayleigh-Ritz method provides upper bounds for the eigenvalues. From the results published it can be concluded that the Lagrangian multiplier method also referred to in the literature as component mode synthesis method or modal synthesis method provides upper bounds for a fixed number of constraints and lower bounds for a fixed number of component modes.

2.2 ***Presentation of the method***

In order to derive the Lagrangian equations of motion first the potential and the kinetic energy expressions are given. For sufficiently small values of the generalised velocities $\dot{q}_i$ the kinetic energy may be approximated by

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\dot{q}_i\dot{q}_j M_{ij} \tag{1}$$

where M_{ij} is the generalised mass coefficient and

$$\dot{q}_i = \frac{\partial q_i}{\partial t} \tag{2}$$

If the generalised coordinates q_i are also small the potential energy can be approximated by

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}q_i q_j K_{ij} \tag{3}$$

where K_{ij} is the generalised stiffness coefficient.

Dowell assumes k constraint conditions in the following form

$$f_r \equiv \sum_{j=1}^{n} q_j a_{rj} = 0, r = 1,2,\ldots,k \tag{4}$$

The Lagrangian of the system is then

$$L = T - U + \sum_{r=1}^{k} \theta_r f_r \tag{5}$$

where θ_r is the Lagrangian multiplier associated with the constraint conditions.

The Lagrangian equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, i = 1,2,..,n \tag{6}$$

$$\frac{\partial L}{\partial \theta_r} = 0, r = 1,2,..,k \tag{7}$$

Expressions (1) and (3) can be simplified if q_i are normal coordinates of the unconstrained system. In this case

$$K_{ij} = \omega_i^2 M_i \delta_{ij}, i, j = 1,..,n \tag{8}$$

$$M_{ij} = M_i \delta_{ij}, i, j = 1,..,n \tag{9}$$

where δ_i is the Kronecker delta and ω_i^2 are the eigenfrequencies of the unconstrained structure. With these results the i th equation of motion for the constrained structure becomes

$$M_i(\ddot{q}_i + \omega_i^2 q_i) - \sum_{s=1}^{k} \theta_s a_{si} = 0 \tag{10}$$

Let the constrained structure vibrate with a frequency ω, the general coordinates q_i and the Lagrangian multiplier θ_s in equation (5) are harmonic functions with frequency ω. Consequently the amplitude coefficients q_i can be expressed in terms of the Lagrangian multipliers θ_s by means of equation (5) as follows

$$q_i = \sum_{s=1}^{k} \frac{\theta_s a_{si}}{M_i(\omega_i^2 - \omega^2)}, i = 1,2, \tag{11}$$

Substitution of the above equation into constraint condition (4) gives

$$D_{rs} = \sum_{i=1}^{n} \frac{a_{si}\theta_r a_{ri}}{M_i(\omega_i^2 - \omega^2)} = 0 \tag{12}$$

The eigenvalue expression can be written as follows

$$\det\{\sum_{i=1}^{n} \frac{a_{si}a_{ri}}{M_i(\omega_i^2 - \omega^2)}\} = 0 \tag{13}$$

Once the eigenvalues of the above equation are known the eigenvectors, i.e. the Lagrangian multipliers θ_s , can be determined.
Substitution of these eigenvectors and associated eigenvalues into equation (4) gives the generalised q_i associated with the eigenvalues of the constrained problem.
Finally, the mode shapes v_i of the constrained structure can be determined as follows

$$v_i = \sum_{j=1}^{n} u_j q_j \tag{14}$$

where u_j are the mode shapes of the unconstrained or unconstrained structure.

For adding supports to the unconstrained structure then coefficients a_{ri} can be determined as follows.
Suppose the structure is required to have *k* point supports, then for a two-dimensional problem

$$v(x_r, y_r) = \sum_{j=1}^{n} u_j(x_r, y_r) q_j, r = 1,2,..,k \tag{15}$$

Comparing the above result with that of equation (4) gives the following identity

$$a_{rj} \equiv u_j(x_r, y_r)$$

Note that the above method can also be applied when several unconstrained structures are coupled.

In matrix form equation (13) can be written as follows

$$\begin{bmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{k1} & . & a_{kn} \end{bmatrix} \begin{bmatrix} \dfrac{1}{M_1(\lambda_1 - \lambda)} & 0 & 0 \\ 0 & . & 0 \\ 0 & 0 & \dfrac{1}{M_n(\lambda_n - \lambda)} \end{bmatrix} \begin{bmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{k1} & . & a_{kn} \end{bmatrix}^T \begin{Bmatrix} \theta_1 \\ . \\ \theta_k \end{Bmatrix} = \cdot$$

or

$$\left[\mathbf{C} \frac{1}{\mathbf{M}_{diag}(\lambda_{diag} - \lambda \mathbf{I})} \mathbf{C}^T \right] \theta = \mathbf{0} \tag{16}$$

2.3 ***Discussion***

The eigenvalue formulation of the Lagrangian multiplier method will present a difficulty when there is a number of eigenvalues, which are common to both the unconstrained and constrained problem, i.e. persistent eigenvalues.

As briefly touched on by Dowell [6], the associated elements have to be removed, such as the associated zero displacements (or rotations) and the persistent eigenvalues.

To solve the eigenvalue equation (16) a dedicated numerical solution procedure is required such as the root finding procedures like the bi-section methods. However, persistent eigenvalues stille pose a problem. One way of locating these eigenvalues is to plot the values of the determinant of equation (16). In cases where infinite values are obtained may point in the direction of existing persistent eigenvalues. Even when this is successful multiple eigenvalues cannot be found. Nowhere in the referenced literature these problems have been sufficiently addressed. Once the eigevalues are established eigenvectors consisting of Lagrangian multipliers can be obtained. Thereafter, the eigenvectors have to be transferred into displacement eigenvectors to plot mode shapes and/or to perform dynamic analyses.

Standard numerical eigenvalue solution procedures are only suitable for the following types of eigenvalue formulations with symmetric matrices

$\mathbf{Au} - \lambda\mathbf{u} = \mathbf{0}$

$\mathbf{Au} - \lambda\mathbf{Bu} = \mathbf{0}$

Another important point that is not mentioned in the vast literature about the component mode synthesis method is the fact that the column vectors in matrix $\mathbf{C}^T$ need to be linear independent, otherwise poor results may be obtained.

Also when constraining points are selected too close to each other the matrix $\mathbf{C}^T$ may become ill conditioned.

Note that unlike the modal constraint method (section 6) the columns are not required to be orthonormal. However an orthonormalisation procedure, like the Gram-Schmidt orthonormalisation procedure, may point in the direction of ill condition of the constraint matrix $\mathbf{C}^T$.

Finally, it should be noted that the Lagrangian multipliers represent the forces (or moments) of constraint acting at the point to achieve zero displacements (or rotations). The elements in equation (16) can be regarded as influence coefficients. It is well known that the Rayleigh-Ritz method provides upper bounds for the eigenvalues.
From the results obtained it can be concluded that the Lagrangian multiplier method also referred to in the literature as component mode synthesis method or modal synthesis method provides upper bounds for a fixed number of constraints and lower bounds for a fixed number of component modes.

3 The Receptance Method

3.1 ***Introduction***
The method of receptances was developed around 1960 by Bishop and Johnson [5]. This method enables calculation of the free vibrational characteristics of a combined system from the characteristics of the component systems.

Wilken and Soedel [23] applied the method to a cylindrical shell with stiffening rings.
The receptance method consists of an evaluation of the responses of the component systems to forces which vary sinusoidally in time and which are applied at the locations where the systems are connected.
These responses, due to a set of sinusoidal forces with amplitudes equal to unity, are the so-called receptances of the component systems. A characteristic feature of the receptance method is that each component system is treated as a separate vibrating system subjected to the forces induced by the other component systems.

3.2 ***Presentation of the receptance method***
When sinusoidally varying forces of frequency ω are applied to two linear undamped systems A and B then

$$\begin{aligned}\mathbf{x}_A &= \boldsymbol{\alpha}_A \mathbf{f}_A \\ \mathbf{x}_B &= \boldsymbol{\alpha}_B \mathbf{f}_B\end{aligned} \tag{17}$$

All quantities are functions of the frequency ω. It is assumed that the displacements and forces of both systems are defined in one global coordinate system. When the two systems are joined and no external forces are applied to the two systems then

$$\begin{aligned}\mathbf{x}_A &= \mathbf{x}_B \\ \mathbf{f}_A &= -\mathbf{f}_B\end{aligned} \tag{18}$$

For the equations (17) and (18) the eigenvalue equation for the combined system is

$$[\boldsymbol{\alpha}_A + \boldsymbol{\alpha}_B]\mathbf{f}_A = \mathbf{0}$$

or

$$\det\{\boldsymbol{\alpha}_A + \boldsymbol{\alpha}_B\} = 0 \tag{19}$$

The values of ω and $\mathbf{f}_A$ which are the solutions of above equation are respectively the natural frequency and the interaction forces mode shapes of the combined system.

The displacement mode shapes can be obtained from equation (17) with ω and $\mathbf{f}_A$ as input.

When the system A is only constrained at k points the eigenvalue expression results from

$$\mathbf{x}_A = \mathbf{0} = \boldsymbol{\alpha}_A \mathbf{f}_A$$

or

$$\det\{\boldsymbol{\alpha}_A\} = 0 \tag{20}$$

A typical expression for the receptance is

$$\alpha_{rs} = \sum_{i=1}^{n} \frac{\phi_i(x_r)\phi_i(x_s)}{M_i(\omega_i^2 - \omega^2)} \tag{21}$$

where based on the notations of the previous section the following identities hold

$$\phi_i(x_r) \equiv a_{ri}$$
$$\phi_i(x_s) \equiv a_{si}$$

3.3 ***Discussions***

A comparison between the Lagrangian multiplier method and the receptance method leads to the conclusion that both methods result in identical constrained eigenvalue equations.
Also the so-called influence coefficients in the previous section are receptances as presented here. The interaction forces are simply the Lagrangian multipliers of the previous section.
From the results obtained it can be concluded that also the receptance method provides upper bounds for a fixed number of constraints and lower bounds for a fixed number of component modes.
See section 2.3 for a discussion about the numerical challenges to extract eigenvaleus and mode shapes when using this method.

4 Weinstein's Determinant of the first type

4.1 ***Introduction***

In the years 1935-1937 Weinstein [3] introduced a method for obtaining lower bounds of eigenvalues that is generally called the method of intermediate problems of the first type. Later, other types of intermediate problems were developed, of which the second type is the most important one.
The scheme of the intermediate problems is as follows.
Given an eigenvalue problem with operator associated with certain boundary conditions, the first step is to find a base problem, namely an eigenvalue problem for which the eigenvalues and eigenfunctions are known. The eigenvalues must be less or equal to the corresponding eigenvalues of the desired problem.
Then a sequence of intermediate problems can be set up that will link the base problem (unconstrained problem) to the desired problem (constrained problem) in such a way that the computed eigenvalues are between those of the base problem and the desired problem.

4.2 ***Weinstein's method in its original form (principles)***

In this section the Weinstein approach will be briefly given in the form that was applied to a plate buckling problem [3]. The desired problem consists of a clamped rectangular plate governed by the following differential equation

$$Au - \lambda u = \nabla^2 u - \lambda u \qquad (22)$$

$$u = \frac{\partial u}{\partial n} = 0 \text{ at boundary } C \tag{23}$$

It is well known that the eigenvalues of $Au = \lambda u$ can also be formulated as the extrema of a variational problem. It will be demonstrated that, for a variational problem where further conditions are added, upper bounds for eigenvalues are obtained. This can be achieved by requiring the function u to lie in a Rayleigh-Ritz manifold of admissible functions.

To find lower bounds, the conditions must be weakened. There is one way of performing this, namely by constituting a suitable base problem so that each of the eigenvalues is a lower bound for the corresponding eigenvalues of the desired problem. An infinite sequence of intermediate problems must be set up to link the base problem eigenvalues to the eigenvalues of the desired problem as has been already mentioned.

For the clamped plate Weinstein selected a simply supported plate as the base problem for which exact eigenelements are known.

The limiting case, i.e. the clamped plate, is achieved by applying boundary conditions in the form of rotational constraints.

4.3 *Formulation of the desired problem*

Assume that the eigenvalue equation of the desired problem (the given problem) is given by

$$A^{(1)}u^{(1)} = \lambda u^{(1)} \tag{24}$$

Operator $A^{(1)}$ has a domain $D^{(1)}$ of the functions $u^{(1)}$ with region $R^{(1)}$ and boundary $C^{(1)}$.

The eigenvalues are denoted by $C^{(1)}$ and the associated eigenfunctions by $u_i^{(1)}$.

When the boundary conditions formed by $C^{(1)}$ are complicated, it could be a difficult task to find admissible functions.

Suppose, e.g., that eigenvalues for a free-free plate supported also at arbitrarily located points, are required. To find admissible functions that will vanish at these arbitrary locations is a nearly impossible task.

Instead of this approach, the method of intermediate problems is applied. In this case a suitable base problem must be defined, i.e. free-free rectangular plate for which approximate eigenvalues and eigenfunctions are known.

4.4 *Formulation of the base problem*

As already indicated in the previous section the desired problem could be solved by using a different conveniently related problem. In other words, a conveniently shaped region R and boundary C are selected, assuming that the region $R^{(1)}$ and boundary $C^{(1)}$ are contained in R .

Also an operator A is selected with a domain D for which it is assumed that $D^{(1)}$ is contained, as a subspace, in D . The domain D consists of functions u defined on R and satisfying C .

It is assumed that the following eigenvalue equation for the base problem is known

$$Au = \lambda u \tag{25}$$

The eigenvalues are denoted by λ_i and the eigenfunctions by u_i , which form a complete orthonormal set in D .
Hence, every element u in D can be expanded as

$$u = \sum_{i=1}^{n} (u, u_i) u_i = \sum_{i=1}^{n} q_i u_i \tag{26}$$

Before the base problem and the desired problem are related the following assumptions have to be made.

Basic assumptions

1. It is assumed that the functions $u^{(1)}$ lying in space $D^{(1)}$ with region $R^{(1)}$ are defined on the entire region R, which contains the region $R^{(1)}$. In other words the eigenfunctions of the base problem must be capable of describing the eigenfunctions of the desired problem. This is a very important notion, which also applies to the Lagrangian method and the receptance method. This has not always been made explicit in literature about these methods.
2. The functions $u^{(1)}$ in $D^{(1)}$ which are solutions of the desired problem $A^{(1)}u^{(1)} = \lambda^{(1)}u^{(1)}$ are restrictions to some set of functions in D with region R to region $R^{(1)}$. Assume that the function u in D is required to vanish at k locations. As already shown previously, the above constraint conditions can be stated as $(u,p_r)=0$ for $r=1,2,...,k$ where p_r are the constraint functions. These conditions restrict the function u to lie in $D^{(1)}$.
3. The self-adjoint, positive-definite operator $A^{(1)}$ with domain $D^{(1)}$ is the restriction of the self-adjoint, positive-definite operator A with domain D to functions in domain $D^{(1)}$. This restriction is established by the projection operator and is associated with the constraint functions p_r.

*4.5 **Relation of the desired problem to the base problem***

From the basic assumption 1 in the previous section it follows that the functions $u^{(1)}$ in $D^{(1)}$ can be extended to D by the addition of new functions v in order to form a complete orthonormal set of functions in D.
Let $D\Theta D^{(1)}$ represent the infinite dimensional orthogonal complement space to $D^{(1)}$.
Every function u in D can then be uniquely expressed as

$$u = u^{(1)} + v \tag{27}$$

where $u^{(1)}$ is in $D^{(1)}$ and v in $D\Theta D^{(1)}$.

As has already been shown in the preceeding sections it is convenient to introduce a set of linear independent (orthonormal) constraint functions p_r, which span the space $D\Theta D^{(1)}$.

Also it has been suggested in the previous section that a function in space D be projected onto the subspace $D^{(1)}$ with the aid of the following projection operator

$$u^{(1)} = u - Pu \tag{28}$$

where

$$\begin{gathered} Pu = \sum_{r=1}^{k} (u, p_r) p_r \\ Pu = v \end{gathered} \tag{29}$$

The functions $A^{(1)}u^{(1)}$ can be identified as projections of the functions Au onto the subspace $D^{(1)}$ orthogonal to $D\Theta D^{(1)}$ and therefore applying the projection operator to Au gives

$$A^{(1)}u^{(1)} = \mathrm{A}u - PAu \tag{30}$$

In the earlier section it is also shown that $A^{(1)} = QA$ which constitutes the relation between the desired operator $A^{(1)}$ and the base operator .
Now with this result the eigenvalue equation for the desired problem can be written as

$$Au - PAu = \lambda u - \lambda Pu \tag{31}$$

The constraint condition requires u to be orthogonal to the constraint functions P_r and hence $Pu = 0$, so that the final eigenvalue equation for the desired problem becomes

$$Au - PAu = \lambda u \tag{32}$$

4.6 Formulation of a finite sequence of intermediate problems

As already mentioned earlier the infinite dimensional subspace $D\Theta D^{(1)}$ must be approximated by successively k intermediate subspaces.
Let M_k denote the finite dimensional subspace of $D\Theta D^{(1)}$ spanned by k constraint functions p_r and denote by $D_k^{(1)}$ the infinite dimensional subspace $D\Theta M_k$ that approximates the subspace $D^{(1)}$.

The following sequence of subspaces can then be set up with the following property

$$D_1^{(1)} \supset D_2^{(1)} \supset \dots\dots \supset D_k^{(1)} \supset \dots\dots \supset D^{(1)} \tag{33}$$

Gould [2] proved that this sequence of subspaces $D_k^{(1)}$ converges for k – ∝ to subspace $D^{(1)}$. Also the projection of D onto $D_k^{(1)}$ must be truncated

$$P_k u = \sum_{r=1}^{k} (u, p_r) p_r \tag{34}$$

Therefore the eigenvalue equation for the k th intermediate problem is as follows

$$Au - P_k Au = \lambda u \tag{35}$$

with eigenvalues $\lambda_{k,1}^{(1)} \leq \lambda_{k,2}^{(1)} \leq \dots\dots$ and associated eigenfunctions $u_{k,1}^{(1)}, u_{k,2}^{(1)}, \dots\dots$ of the k th intermediate problem.

It can be proved that the eigenvalues $\lambda_{k,1}^{(1)}, \lambda_{k,2}^{(1)}, \dots.$ converge with increasing k to $\lambda_1^{(1)}, \lambda_2^{(1)}, \dots\dots$ respectively since the sequence of subspaces $D_k^{(1)}$ converges to $D^{(1)}$.

In the next section the Weinstein determinant of the first type will be established for arbitrary constraint functions.

4.7 *The Weinstein determinant and non-persistent eigenvalues*

Based on equation (34) the following expression can be written

$$P_k Au = \sum_{r=1}^{k} (Au, p_r) p_r = \alpha_1 p_1 + \dots + \alpha_k p_k$$

The desired eigenvalue equation (35) can be written as follows

$$Au - \lambda u = \alpha_1 p_1 + \alpha_2 p_2 + \dots. + \alpha_k p_k \tag{36}$$

where

$$(u, p_s) = 0, s = 1, 2, \dots., k$$

The scalars α_r are not given scalars but depend on the unknown function u . To establish the magnitude of α_r , the inner products with p_s are formed, resulting in the following equations

$$(PAu,p_s)=(Au,p_s)=\sum_{r=1}^{k}\alpha_r(p_r,p_s), s=1,2,....,k \tag{37}$$

If the constraint functions p_s form a complete orthonormal set then the following simple results is obtained

$$\alpha_r = (Au, P_r) \tag{38}$$

This is a very important result and together with equation (36) is the most important starting point for establishing the modal constraint method.
If the eigenvalue λ is not in the spectrum of operator A, or in other words is a non-persistent eigenvalue then equation (36) can be written as follows

$$u=\sum_{r=1}^{k}\alpha_r R_\lambda p_r \tag{39}$$

where R_λ is the resolvent operator defined as

$$R_\lambda = [A - \lambda I]^{-1} \tag{40}$$

Using the orthogonality relations $(u, p_s) = 0$ the following set of equations are obtained

$$0=(u,p_s)=\sum_{r=1}^{k}\alpha_r(R_\lambda p_r,p_s), s=1,2,....,k \tag{41}$$

The above eigenvalue equation has a nontrivial solution for $\alpha_1,.....,\alpha_k$ if and only if

$$W(\lambda)=\det\{(R_\lambda p_r,p_s)\}=0 \tag{42}$$

The function W (λ) is called Weinstein's determinant of the first type for non-persistent eigenvalues.

Applying the resolvent operator R_λ gives

$$W(\lambda)=\det\{\sum_{i=1}^{n}\frac{(p_r,u_i)(p_s,u_i)}{\lambda_i-\lambda}\}=0 \tag{43}$$

where λ_i and u_i are associated with the base problem.
In the next chapter the particular choice for the constraint function p_r will be discussed.

5 The particular choice of constraint functions

5.1 *Introductory remarks*

In this chapter the particular choice of the constraint functions will be discussed. As already mentioned these constraint functions are related to the type of constraint. There are three main types of constraint that can be used, i.e. point supports, line supports and area supports. In this paper only point supports will be discussed.

A line or area support can be approximated by *k* point supports that constitute *k* constraint functions p_s and therefore a sequence of *k* intermediate problems.

When expressions for the constraint functions are found for point supports it is possible to write Weinstein's equation (43) into an eigenvalue equation for non-persistent eigenvalues for the desired problem having *k* points.

After this result this eigenvalue problem is transformed into an eigenvalue equation that will allow persistent (common to both base and desired problem) eigenvalues, i.e. the modal constraint method, see section 6.

5.2 *Particular choice of constraint functions for adding point supports to the base problem*

In practical calculations the complement region R - $R^{(1)}$ and the boundary $C^{(1)}$ may have complicated shapes and must, therefore, be decomposed into a series of constituents. These components generate a theoretically infinite chain of constraints of the type given in equation (44), which must be approximated by a finite number *k* of these constraints. Therefore, also *k* constraint functions p_s exist. The successive application of these *k* approximating constraint relations generates what is previously called the *k* th intermediate problem. In this section, a particular choice for the constraint functions p_1,p_2,...,p_k will be established, which corresponds to *k* constraint conditions.

These constraint or continuity conditions constrain the function u on the region R - $R^{(1)}$ with the inclusion of boundary $C^{(1)}$, if necessary.

As already discussed these constraints will take the form of an expression between the generalised coordinates q_i

$$h(q_1, q_2, ..., q_n) = 0 \tag{44}$$

Taylor expansion about equilibrium, neglecting higher order terms (only small vibrations), gives the following linear function

$$h(q_1, q_2, ..., q_n) = \sum_{i=1}^{n} \frac{\partial h}{\partial q_i} |_0 \; q_i = 0 \tag{45}$$

or

$$h(q_1,q_2,....,q_n)=\sum_{i=1}^{n}c_i q_i=0 \tag{46}$$

If there are k distinct constraint conditions or linear independent constraint relations then k equations like equation (46) can be set up

$$h_l(q_1,q_2,....,q_n)=\sum_{i=1}^{n}c_{li}q_i=0, l=1,2,...,k \tag{47}$$

Note that the desired problem has in this case (n - k) degrees of freedom and therefore k eigenvalues equal to zero.

A one-to-one correspondence of equations (47) with constraint functions p_s will be established using orthogonality condition $(u, p_s) = 0$. Because each constraint function p_s is in the space D, which is spanned by the eigenfunctions u_i, the following expansion can be defined

$$p_s=\sum_{i=1}^{n}b_{si}u_i, s=1,2,....,k \tag{48}$$

where b_{si} are the expansion coefficients.
Taking the scalar product $(u, p_s) = 0$ with equation (26) in mind then yields

$$(u,p_s)=\sum_{i=1}^{n}b_{si}q_i=0 \tag{49}$$

Now the one-to-one correspondence between equation (47) and (49) can be observed. Equation (48) can then be written as

$$p_s=\sum_{i=1}^{n}c_{si}u_i \tag{50}$$

The finite point set constraint decomposition consists of representing the constraint conditions for the region R - $R^{(1)}$ and/or boundary $C^{(1)}$ by a finite number of points.
Assume for sake of simplicity a two-dimensional problem where the finite points are located at coordinates x_s, y_s. The function u is required to vanish at each of these points

$$u(x_s,y_s)=\sum_{i=1}^{n}u_i(x_s,y_s)q_i, s=1,2,....,k \tag{51}$$

If the above equation is compared with equation (49) then the one-to-one correspondence leads to the following identity

$$u_i(x_s,y_s)\equiv c_{si} \tag{52}$$

Note that also rotations may be required to vanish at certain points.
In this case (normal to the boundary in x direction)

$$\frac{\partial u(x_s, y_s)}{\partial x} \equiv c_{si} \tag{53}$$

It is pointed out that also several substructures could be coupled at a number of discrete locations. The constraints take then the form where the differential displacements are required to be zero at the coupling locations [9] and [10].

5.3 Weinstein's determinant for a particular choice of constraint functions

Up to now the constraint function p_r in Weinstein's determinant could be arbitrarily chosen. In the previous section the relation is demonstrated between adding point supports to a base problem and a particular expression for these constraint functions.
In this section the starting expression is equation (43)

$$\det\{\sum_{i=1}^{n} \frac{(p_r, u_i)(p_s, u_i)}{\lambda_i - \lambda}\} = 0 \tag{54}$$

Substituting expression (50) for p_j in the above equation renders an eigenvalue expression for non-persistent eigenvalues ($\lambda_i \neq \lambda$)

$$\det\{\sum_{i=1}^{n} \frac{c_{ri} c_{si}}{\lambda_i - \lambda}\} = 0 \tag{55}$$

The above equation is identical to the equations obtained by either the receptance method [5] or the Lagrangian multiplier method [6] with the following identities

$$c_{ri} \equiv a_{ri}$$
$$c_{si} \equiv a_{si}$$

With this result the receptance method and the Lagrangian multiplier method is linked to Weinstein's method for intermediate problems of the first type.

6 The Modal Constraint method originating from a differential operator approach

In this section the Weinstein classical theory will be extended to arrive at the formulation of the modal constraint method.

The fundamental idea is to expand the scalars $\alpha_r, r = 1,2,....,k$ in terms of the generalised coordinates q_i so that the eigenfunctions are retained without an additional step.

It is recalled that

$$\alpha_r = (Au, p_r) \tag{56}$$

which is appropriate for the special case when constraint functions are orthonormal. In most cases this orthonormality is not guaranteed and an orthonormalisation procedure must be applied to constraint functions p_i (e.g. the Gram-Schmidt orthonormalisation procedure).

Making use of equations (50) and (52) the following expression with known quantities λ_i, c_{ji} and the generalised coordinates q_i is obtained

$$\alpha_r = (Au, p_r) = \sum_{i=1}^{n} c_{ri}\lambda_i q_i, r = 1,2,....,k \tag{57}$$

In matrix form this equation can be written as

$$\boldsymbol{\alpha} = \mathbf{C}\boldsymbol{\lambda}_{diag}\mathbf{q} \tag{58}$$

where

$\mathbf{C}^T = \begin{bmatrix} c_{11} & . & c_{k1} \\ . & . & . \\ c_{1n} & . & c_{kn} \end{bmatrix}$, which is called the constraint matrix.

Equation (57) can also be written in matrix form

$$\mathbf{q} = [\boldsymbol{\lambda}_{diag} - \lambda\mathbf{I}]\mathbf{C}^T\boldsymbol{\alpha} \tag{59}$$

Substituting equation (59) into (60) gives

$$\mathbf{q} = [\boldsymbol{\lambda}_{diag} - \lambda\mathbf{I}]\mathbf{C}^T\mathbf{C}\boldsymbol{\lambda}_{diag}\mathbf{q} \tag{60}$$

After rearranging

$$[[\mathbf{I}-\mathbf{C}^T\mathbf{C}]\boldsymbol{\lambda}_{diag}-\lambda\mathbf{I}]\mathbf{q}=\mathbf{0} \tag{61}$$

Note that the left hand expression in the above matrix eigenvalue expression is asymmetric. To arrive at symmetric matrices, equation (61) is again rearranged [6]

$$[[\mathbf{I}-\mathbf{C}^T\mathbf{C}]-\lambda\boldsymbol{\lambda}_{diag}]\overline{\mathbf{q}}=\mathbf{0};\mathbf{q}=\boldsymbol{\lambda}_{diag}^{-1}\overline{\mathbf{q}} \tag{62}$$

The above expression allows also persistent eigenvalues and multiple eigenvalues to be evaluated with standard numerical eigenvalue methods.
Note that the constraint matrix $\mathbf{C}^T$ needs to be orthonormalised row by row, which is due to the requirement that the constraint functions p_i must be orthonormal. This can be achieved by applying the Gram-Schmidt orthonormalisation procedure. The short notation of equation (62) takes the form of $[\mathbf{A}-\lambda\mathbf{B}]\overline{\mathbf{q}}=\mathbf{0}$ with two symmetric matrices. This eigenvalue expression can be solved by standard numerical eigenvalue extraction procedures.

7 The modal constraint method in terms of energy: Lagrangian multipliers

Following the development given in [7] the energy functional for the constrained problem is

$$E=\frac{1}{2}\int_A(Lu)u\,dA+\frac{1}{2}\rho\int_A(\dot{u},\dot{u})dA-\sum_{r=1}^{k}W_r \tag{63}$$

where L is the differential operator, $\dot{u}=\dfrac{\partial u}{\partial t}$ and A is the domain of the problem.

The constraint energy that has to be subtracted from the energy of the unconstrained problem is

$$W_r=\alpha_r\sum_{i=1}^{n}c_{ri}q_i \tag{64}$$

where α_j are the Lagrangian multipliers.
A one-to-one correspondence between equations (57) and (64) leads to the conclusion that the constraint forces to enforce the constraint are similar to

$$\alpha_r=(Au,p_r) \tag{65}$$

where (see equation (50))

$$p_r=\sum_{i=1}^{n}c_{ri}u_i \tag{66}$$

Taking the first variation of equation (63) leads, after some elaboration, to the same eigenvalue equation for the constrained problem as equation (62).

8 Practical example: four point supported square plate

In this section a specific example will be given where persistent eigenvalues will appear. It will be shown that this will not pose any problems to extract these eigenvalues contrary to the methods, which have similar expressions as equation (55).
The example consists of a four pointed supported square plate with three supports located at the corners of the plate and one located at 0.15 of its diagonal length, see Figure 1.

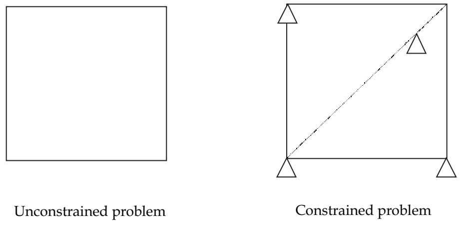

Figure 1. Vibrating square plate: unconstrained problem and constrained problem

First let us introduce the formulation of the unconstrained problem, i.e. a completely free vibrating square plate.
In [8] expressions for the unconstrained problem are given. It is also shown there how to establish eigenvalues and eigenvectors based on beam functions of a completely free vibrating rectangular plate. See Figure 2.
Note that also the three rigid body modes have to be added to the set of elastic mode shapes, denoted by $\psi_i; i = 1,.., n$.

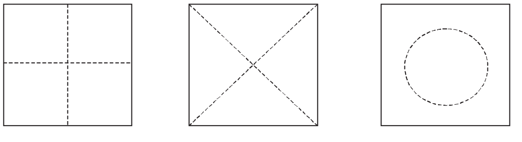

Figure 2. Unconstrained problem: completely free vibrating plate: first three elastic mode shapes

The eigenvalue equation for the constrained problem, i.e. where point supports are added is given below

$$[[\mathbf{I} - \mathbf{C}^T\mathbf{C}] - \lambda\boldsymbol{\lambda}_{diag}^{-1}]\bar{\mathbf{q}} = \mathbf{0} \tag{67}$$

The elements of the orthonormalised constraint matrix $\mathbf{C}^T$ are: $c_{ri} = \psi_i(\xi_r, \eta_r)$ where there are k constraint relations being the four point supports (ξ_r, η_r), $r = 1,\dots,4$ and i is the number of modes.

The following results are obtained.

Table 1. Eigenvalues of the four point supported rectangular plate: $\omega a^2 \sqrt{\rho / D}$.

	First	**second**	**third**
Unconstrained problem	13.373	**19.496**	23.784
Constrained problem	8.596	15.646	**19.496***)

*) persistent eigenvalue, i.e. common to both the unconstrained problem and the constrained problem

This can be understood by examining the mode shapes of the constrained problem, see Figure 3.

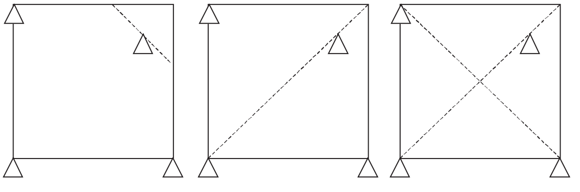

Figure 3. Constrained problem: first three mode shapes: persistent eigenvalue.

The second mode shape of the unconstrained problem is equal to the third mode shape of the constrained problem because all point supports are located at the nodal lines of the second mode shape. The requirement of zero displacements is here already satisfied and therefore this mode shape and its associated eigenvalue will be retained, i.e. persistent.
Using Weinstein's determinant, Lagrangian multiplier method or receptance method will present a problem because the persistent eigenvalue will make the dominator of expression (55) zero and the value of the determinant infinitely large. When this is anticipated the considered eigenvalues should be removed from the range of constrained eigenvalues. This is not always a straightforward affair.

9 Conclusions

The Lagrangian multiplier method and the receptance method have similar constrained eigenvalue expressions to Weinstein's determinant for intermediate problems of the first type. These methods do not allow persistent eigenvalues to be present, i.e. those eigenvalues that are common to both the unconstrained problem and the constrained problem.
To solve the resulting eigenvalue equations dedicated numerical solution procedures are required such as the root finding procedures like the bi-section methods. However, persistent eigenvalues stille pose a problem. One way of locating these eigenvalues is to plot the values of the determinant of the eigenvalue equations. In cases where infinite values are obtained may point in the direction of existing persistent eigenvalues. Even when this is successful multiple eigenvalues cannot be found. These problems have not been sufficiently addressed in the referenced literature. Once the eigevalues are established eigenvectors consisting of Lagarangian multipliers can be obtained. Thereafter, the eigenvectors have to be transferred into displacement eigenvectors to plot made shapes and/or to perform dynamic analyses.

In addition it should be noted that the Lagrangian multipliers represent the forces (or moments) of constraint acting at the point to achieve zero displacements (or rotations).

The elements in the resulting eigenvalue equations can be regarded as influence coefficients. It is well known that the Rayleigh-Ritz method provides upper bounds for the eigenvalues.
From the results obtained it can be concluded that the Lagrangian multiplier method also referred in the literature component mode synthesis method or modal synthesis method provides upper bounds for a fixed number of constraints and lower bounds for a fixed number of component modes.

Comparing the Lagrangian multiplier method and the receptance method reveals that influence coefficients have similar expressions as the receptances.

Weinstein's method for intermediate problems of the first type allow the development of the modal constraint method by realising that the Lagrangian multipliers can be expended into known quantities, i.e. they are inner products of the unconstrained operator with constraint functions, which in turn are related to the physical constraints.
The resulting constrained eigenvalue expression can be solved with standard numerical eigenvalue routines. Also zero modal displacements (or rotations) do not pose a problem.

In the literature about Lagrangian multiplier methods and receptance methods no special attention is paid to precautions needed to avoid an ill-conditioned state of the constraints matrices. This can be diagnosed by applying the Gram-Schmidt orthonormalisation procedure, which is a requirement for the modal constraint method. In other words, no remarks can be found in the vast literature about the component mode synthesis method that the column vectors in matrix need to be linearly independent, otherwise poor results may be obtained. Also when constraining points are selected too close to each other the matrix may become ill conditioned.

The results obtained by various authors show that increasing the number of constraints for a fixed number of terms will result in lower bounds for the constrained eigenvalues whereas increasing the number of terms for a fixed number of constraints will result in upper bounds as is well-known for applying the classical Rayleigh-Ritz method. This hypothesis can be proved to be true but in view of the available space it will be an issue for a separate paper.

It is pointed out that coupling unconstrained subsystems may also be treated by the methods displayed. These problems merely require the differential modal displacements of the two subsystems to be zero.

Finally, it is mentioned that also buckling of constrained linear mechanical systems can be treated with the modal constraint method as buckling of structures can be seen as an eigenvalue problem.

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