Linear elastic lateral buckling (including shear deformation) and linear elastic lateral torsional buckling of composite beams – An analytical engineering approach

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This article outlines a concise and complete versatile novel analytical framework to quickly and effectively assess lateral shear (knik) and lateral torsional buckling (kip) of composite beams build of sections with different lengths. The adjective "composite" includes dissimilar beam cross-sections (beam profiles) and distinct beam materials. The major aim is to provide engineers with an accurate, simple but elegant, practically applicable and analytical beam buckling tool based on advanced solid theoretical propositions or background. The justification of the analytical theory is supplied by demonstration of executed numerical tests (i.e. finite element method), which affirm the validity of the proposed analytical buckling models adapted to the considered beam buckling cases.

Key words: Lateral buckling, lateral torsional buckling, composite beam, analytical formulation, shear deformation, linear elasticity

1 Introduction

Buckling means the loss of stability of an equilibrium configuration of a structure. Elastic buckling is a state at which the structure loses its stability and large elastic deflections will start developing rapidly. It is normally associated with the minimum eigenvalue of the perfect structure, i.e. often referred to as classical buckling (DNV 2015). In this article attention is confined to linear elastic composite beam buckling (especially lateral and lateral torsional buckling phenomena). An abbreviated and lucid literature survey and comprehensive discussion about lateral and lateral torsional buckling of structural members is rendered in the dissertation of Raven (Raven 2006) and the article of Van der

Put (Put, van der 2008). It should be highlighted that the adjective "composite" refers to multiple beam cross-sections and beam materials.

Approximate solution methods are frequently encountered and used in engineering design practice for tackling beam buckling problems. In particular, summation theorems (i.e. Föppl-Papkovich theorem) are appropriate and advantageous because of their robustness, accuracy, easiness and algebraic simplicity (Tarnai 1995). It should be kept in mind that there is similarity or analogy with parallel and serial analytical approaches of spring systems, electric systems or hydraulic systems (the so-called analogies).

The key goal of this article is to address respectively formulate an effective analytical tool for reliable (engineering) estimation or approximation of the linear elastic lateral shear buckling force and linear elastic lateral torsional buckling moment of composite beams made of non-overlapping parts based on the indicated summation theorem and corresponding well-established structural elastic stability notions (Petersen 2013), (Timoshenko 1985). A succinct but detailed exposition of the previous stated matters and issues is offered in the subsequent sections.

2 Preliminaries and requisites

Global Cartesian (X, Y, Z components) and local Cartesian (x, y, z components) right-handed coordinate systems are used. The former identify the composite beam orientation, beam location and beam external loads, while the latter define attached geometrical (shape and size) properties of the homogenous beam cross-sections. It is henceforth assumed that the origin of the local coordinate system is located at the centroid C of the cross-sectional area. In addition, the initial (undeformed) geometry of the composite beam is a straight line. The following beam cross sectional quantities (properties) are employed (Hartsuijker and Welleman 2007).

$$A = \int_A dA, \quad I_{yy} = \int_A y^2 dA, \quad I_{yz} = I_{zy} = \int_A y\, z\, dA = 0, \quad I_{zz} = \int_A z^2 dA, \quad I_t \tag{1}$$

where

A	is the cross sectional area,
$I_{yy}, I_{yz}, I_{zy}, I_{zz}$	are called second moments of area,
I_t	is the torsion constant.

Note 1. The centroid C is defined as that point of an area A for which the static moments of area are zero when the origin of the local $x\ y\ z$ coordinate system is chosen there, i.e. $S_y = \int_A y\, dA = 0$ and $S_z = \int_A z\, dA = 0$.
The normal centre NC is specified as that point of the cross-sectional area where the resultant of all normal stresses due to extension has its point of application.

Note 2. In a homogenous (single material) cross-section, the centroid C and normal centre NC coincide.

Note 3. The adopted assumption $I_{yz} = I_{zy} = 0$ implies that the selected beam cross-section has at least two lines of symmetry.

Note 4. I_{yy} and I_{zz} are commonly indicated as the moments of inertia and I_{yz} , I_{zy} are known as the product of inertia.

The Equations (1) have been applied to the cross-sections in Figure 1. The results are shown below.

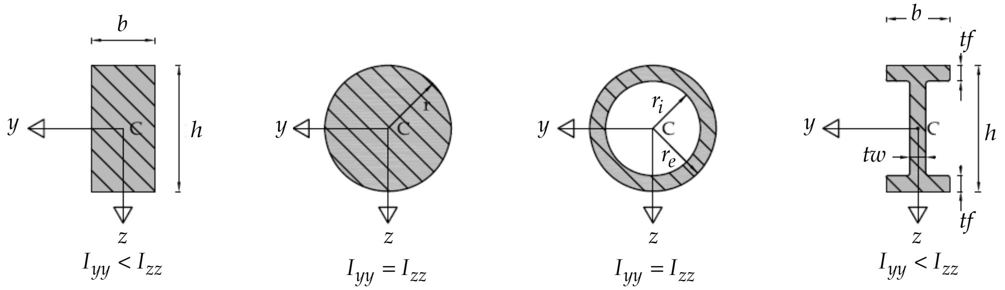


Figure 1: Beam cross-sections

Rectangular cross-section: $A = bh,\quad I_{yy} = \frac{1}{12} b^3 h,\quad I_{zz} = \frac{1}{12} bh^3,\quad I_{yz} = 0,\quad I_t = \alpha \frac{1}{3} hb^3$,

Solid circular cross-section: $A = \pi r^2,\quad I_{yy} = \frac{1}{4}\pi r^4,\quad I_{zz} = \frac{1}{4}\pi r^4,\quad I_{yz} = 0,\quad I_t = \frac{1}{2}\pi r^4$,

Hollow circular cross-section: $A = \pi(r_e^2 - r_i^2),\quad I_{yy} = \frac{1}{4}\pi(r_e^4 - r_i^4),\quad I_{zz} = \frac{1}{4}\pi(r_e^4 - r_i^4),$

$$I_{yz} = 0,\quad I_t = \frac{1}{2}\pi(r_e^4 - r_i^4)\ ,$$

I cross-section: $I_t = \eta \frac{1}{3}(b\, t_f^3 + (h - t_f) t_w^3 + b\, t_f^3)$,

where

A	is the area of the cross-section,
E	is the modulus of elasticity or Young's modulus,

EI_{yy}	is the bending stiffness in the $x\,y$ plane,
EI_{zz}	is the bending stiffness in the $x\,z$ plane,
$G = \dfrac{E}{2(1+\nu)}$	is the shear modulus,
ν	is the lateral contraction coefficient or Poisson's ratio,
GI_t	is the torsional stiffness.

3 Linear elastic lateral buckling (including shear deformation) (knik)

*3.1 **Model geometry***

The typical model geometry is depicted in Figure 2. The following mechanical boundary (displacement and load) conditions are implemented (prescribed).

Boundary 1. $u_X = 0,\ u_Y = 0,\ u_Z = 0$

$F_X = 1$ (applied at centroid)

Boundary 2. $u_Y = 0,\ u_Z = 0$

$F_X = -1$ (applied at centroid)

Note 1. A set of balanced forces is obligatory in order to preserve equilibrium of the composite beam in the analytical model configuration.

Note 2. The prescribed boundary displacement constraints prevent rigid body motions (circumvent singularities in the numerical (finite element) solution process).

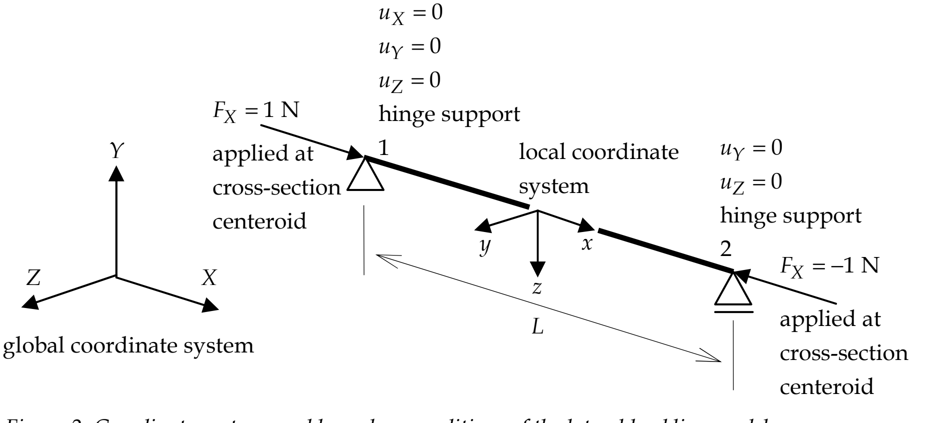


Figure 2: Coordinate systems and boundary conditions of the lateral buckling model

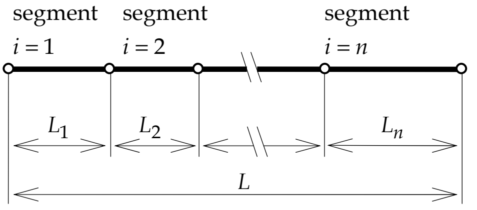


Figure 3: Composite model geometry

3.2 *Analytical model*

The composite beam with length L is made of segments i with corresponding length L_i (Fig. 3) and each segment (section) i has the accompanying attributes Young's modulus E_i, shear modulus G_i, shear deformation coefficient k_{si} (Pilkey 2002)(Pilkey 2005), cross-section area A_i and second moment of area I_i. Furthermore, the cross-section of each segment i fulfils the condition $I_{yyi} < I_{zzi}$, which implies that $I_i = I_{yyi}$. Consequently, the following relation holds.

$$L = \sum_{i=1}^{n} L_i \tag{2}$$

with

n is the total number of attained segments (sections).

The engineering estimate of the buckling length $L_{\text{buc}\,i}$ of segment i is given by the expression

$$L_{\text{buc}\,i} = k_{\text{buc}}\, L_i \tag{3}$$

where,

k_{buc} is the buckling length factor, which is identical for each segment in the particular composite beam arrangement.

$$k_{\text{buc}} = \sqrt[2]{\frac{k_{\text{bc}}^2 L^2}{\sum_{i=1}^{n} L_i^2}} = \sqrt[2]{\frac{k_{\text{bc}}^2 \left(\sum_{i=1}^{n} L_i\right)^2}{\sum_{i=1}^{n} L_i^2}} \tag{4}$$

where,

k_{bc} is the buckling length factor of the composite beam, dependent on the mechanical boundary conditions (supports). In this specific case of a composite beam with span L and supported by two hinges at the edges $k_{\text{bc}} = 1$.

It is emphasized that the buckling length of the composite beam depends only on the boundary conditions and thus not on cross-sectional properties. Substitution of Equation 4 into Equation 3 produces the following identity.

$$L_{\text{buc }i} = \sqrt[2]{\frac{(\sum_{i=1}^{n} L_i)^2}{\sum_{i=1}^{n} L_i^2}} L_i \tag{5}$$

Subsequently, the linear elastic lateral shear buckling force $F_{\text{buc }i}$ of segment i is universally elaborated as shown in Equation 6a. Note that shear deformation is also incorporated, see (Petersen 2013), (Timoshenko 1985).

$$\frac{1}{F_{\text{buc }i}} = \frac{1}{\frac{1}{1+\frac{(\frac{\pi^2 E_i I_i}{k_{\text{buc}}^2 L_i^2})}{k_{\text{s }i} G_i A_i}} \frac{\pi^2 E_i I_i}{k_{\text{buc}}^2 L_i^2}} = \frac{1}{\frac{\pi^2 E_i I_i}{k_{\text{buc}}^2 L_i^2}} + \frac{1}{k_{\text{s }i} G_i A_i} = \frac{1}{F_{\text{buc lateral }i}} + \frac{1}{F_{\text{buc shear }i}} \tag{6a}$$

where,

$F_{\text{buc }i}$	is the linear elastic lateral shear buckling force of segment i,
$F_{\text{buc lateral }i}$	is the linear elastic lateral buckling force of segment i,

$$F_{\text{buc lateral }i} = \frac{\pi^2 E_i I_i}{k_{\text{buc}}^2 L_i^2} \tag{6b}$$

$F_{\text{buc shear }i}$ is the linear elastic shear buckling force of segment i.

$$F_{\text{buc shear }i} = k_{\text{s }i} G_i A_i \tag{6c}$$

Recalling from literature that the Föppl-Papkovich theorem (Tarnai 1995) is valid for this case, the following lateral shear buckling system force equation is stated in general format.

$$\frac{1}{F_{\text{buc sys}}} = \sum_{i=1}^{n} \frac{1}{F_{\text{buc }i}} \tag{7}$$

where,

$F_{\text{buc sys}}$	is the linear elastic lateral shear buckling force of the composite beam (also denoted as system),
n	is the total number of considered segments.

The aforementioned formulae devise the complete linear elastic lateral shear buckling analytical model.

4 Numerical verification (case studies) knik

Three case studies (KNIK 1, KNIK 2 and KNIK 3) are conducted and concisely presented in order to examine the validity and soundness of the proposed analytical approach by comparison with numerical results obtained by the finite element method (Bathe 2016), (Zienkiewicz, Taylor and Fox 2014).

Case study KNIK 1-1

Case KNIK 1-1 considers a wooden beam of one segment (Fig. 4).

The properties are L = 3000 mm, L_1 = 3000 mm, $k_{\text{buc}} = 1$, $k_{\text{bc}} = 1$, $F_X = \pm 1$ N, E_1 = 4500 N/mm², G_1 = 1731 N/mm², $\nu_1 = 0{,}3$, $b \times h$ = 60 mm x 160 mm, A_1 = 9600 mm², $I_1 = I_{yy1} = 0{,}288 \cdot 10^7$ mm⁴, $k_{s1} = 0{,}842105$.

The obtained results are $F_{\text{buc sys analytical}}$ = 14214 N, $F_{\text{buc sys numerical}}$ = 14221 N (Fig. 5). The deviation is computed as $\frac{F_{\text{buc sys numerical}}}{F_{\text{buc sys analytical}}} 100\% - 100\% = +0{,}049\ \%$.

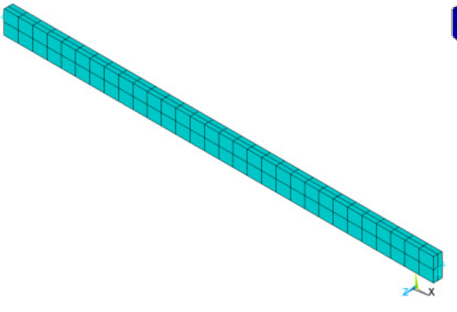


Figure 4: Geometry of case KNIK 1-1

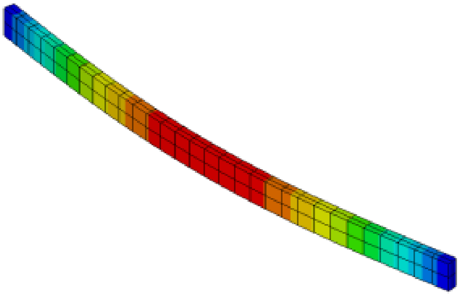


Figure 5: Buckling shape of case KNIK 1-1

Case study KNIK 1-2

Case KNIK 1-2 considers a composite wooden beam of two segments (Fig. 6). The properties are L = 3000 mm, L_1 = 1500 mm, L_2 = 1500 mm, $k_{\text{buc}} = \sqrt{2}$, $k_{\text{bc}} = 1$, $F_X = \pm 1$ N,

E_1 = 4500 N/mm², G_1 = 1731 N/mm², $\nu_1 = 0{,}3$,

E_2 = 4500 N/mm², G_2 = 1731 N/mm², $\nu_2 = 0{,}3$,

b x h = 60 mm x 160 mm, A_1 = 9600 mm², $I_1 = I_{yy1}$ = 0,288·10⁷ mm⁴, k_{s1} = 0,842105,

b x h = 30 mm x 80 mm, A_2 = 2400 mm², $I_2 = I_{yy2}$ = 0,18·10⁶ mm⁴, k_{s2} = 0,842105.

The results are $F_{\text{buc sys analytical}}$ = 1673 N, $F_{\text{buc sys numerical}}$ = 1443 N (Fig. 7). The deviation is -13,74 %.

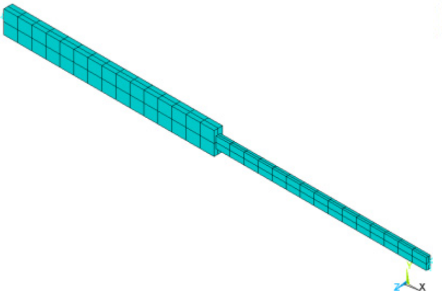


Figure 6: Geometry of case KNIK 1-2

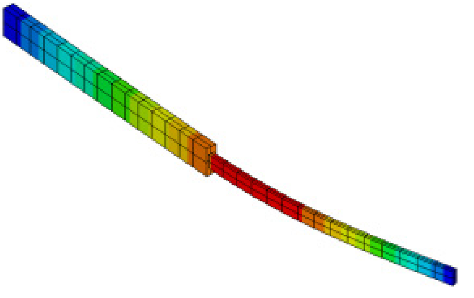


Figure 7: Buckling shape of case KNIK 1-2

Case study KNIK 1-3

Case KNIK 1-3 considers a composite wooden beam of three segments (Fig. 8).

The properties are L = 3000 mm, L_1 = 1000 mm, L_2 = 1000 mm, L_3 = 1000 mm,

$k_{\text{buc}} = \sqrt{3}$, k_{bc} = 1, F_X = ±1 N,

E_1 = 4500 N/mm², G_1 = 1731 N/mm², ν_1 = 0,3,

E_2 = 4500 N/mm², G_2 = 1731 N/mm², ν_2 = 0,3,

E_3 = 4500 N/mm², G_3 = 1731 N/mm², ν_3 = 0,3,

b x h = 60 mm x 160 mm, A_1 = 9600 mm², $I_1 = I_{yy1}$ = 0,288·10⁷ mm⁴, k_{s1} = 0,842105,

b x h = 30 mm x 80 mm, A_2 = 2400 mm², $I_2 = I_{yy2}$ = 0,18·10⁶ mm⁴, k_{s2} = 0,842105,

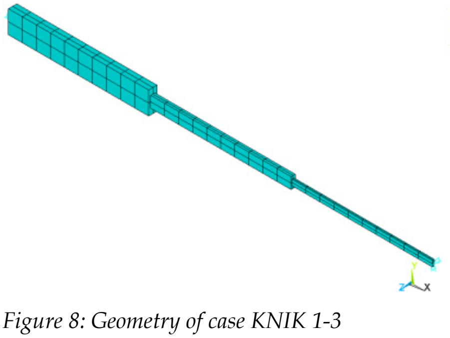


Figure 8: Geometry of case KNIK 1-3

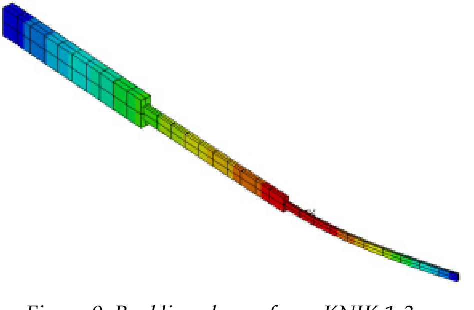


Figure 9: Buckling shape of case KNIK 1-3

b x h = 15 mm x 40 mm, A_3 = 600 mm², $I_3 = I_{yy3}$ = 11250 mm⁴, k_{s3} = 0,842105.

The results are $F_{\text{buc sys analytical}}$ = 156 N and $F_{\text{buc sys numerical}}$ = 161 N (Fig. 9). The deviation is +3,11 %.

Case Study KNIK 2-1

Case KNIK 2-1 considers a steel beam of one segment (Fig. 10).

The properties are L = 3000 mm, L_1 = 3000 mm, k_{buc} = 1, k_{bc} = 1, $F_X = \pm 1$ N,

$E_1 = 2{,}1 \cdot 10^5$ N/mm², G_1 = 80769 N/mm², ν_1 = 0,3,

IPE 200, A_1 = 2725 mm², $I_1 = I_{yy1} = 0{,}142 \cdot 10^7$ mm⁴, k_{s1} = 0,388499.

The obtained results are $F_{\text{buc sys analytical}}$ = 328263 N and $F_{\text{buc sys numerical}}$ = 326493 N (Fig. 11). The deviation is -0,539 %.

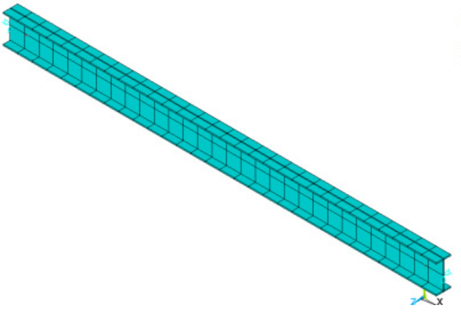


Figure 10: Geometry of case KNIK 2-1

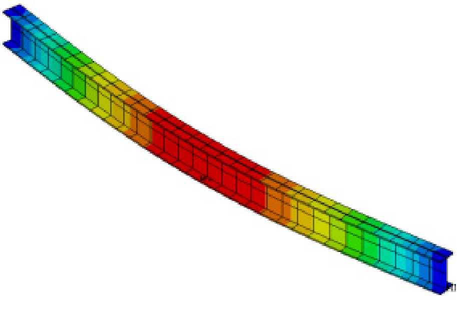


Figure 11: Buckling shape of case KNIK 2-1

Case Study KNIK 2-2

Case KNIK 2-2 considers a composite steel beam of two segments (Fig. 12). The properties are L = 3000 mm, L_1 = 1500 mm, L_2 = 1500 mm, $k_{\text{buc}} = \sqrt{2}$, k_{bc} = 1, $F_X = \pm 1$ N,

$E_1 = 2{,}1 \cdot 10^5$ N/mm², G_1 = 80769 N/mm², ν_1 = 0,3,

$E_2 = 2{,}1 \cdot 10^5$ N/mm², G_2 = 80769 N/mm², ν_2 = 0,3,

IPE 200, A_1 = 2725 mm², $I_1 = I_{yy1}$ = 0,142·107 mm⁴, k_{s1} = 0,388499,

IPE 140, A_2 = 1601 mm², $I_2 = I_{yy2}$ = 448461 mm⁴, k_{s2} = 0,386579.

The results are $F_{\text{buc sys analytical}}$ = 157757 N and $F_{\text{buc sys numerical}}$ = 147745 N (Fig. 13) The deviation is -6,346 %.

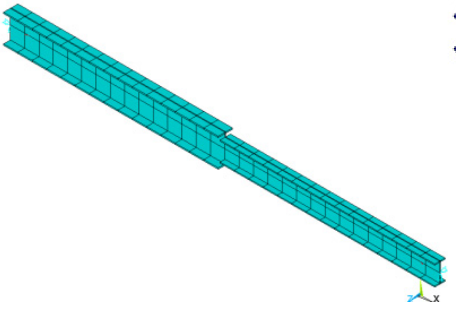


Figure 12: Geometry of case KNIK 2-2

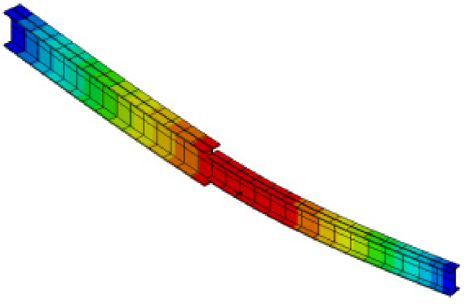


Figure 13: Buckling shape of case KNIK 2-2

Case Study KNIK 2-3

Case KNIK 2-3 considers a composite steel beam of 3 segments (Fig. 14).

The properties are L = 3000 mm, L_1 = 1000 mm, L_2 = 1000 mm, L_3 = 1000 mm,

$k_{\mathrm{buc}} = \sqrt{3}$, k_{bc} = 1, F_X = ±1 N,

E_1 = 2,1·10^5 N/mm^2, G_1 = 80769 N/mm^2, ν_1 = 0,3,

E_2 = 2,1·10^5 N/mm^2, G_2 = 80769 N/mm^2, ν_2 = 0,3,

E_3 = 2,1·10^5 N/mm^2, G_3 = 80769 N/mm^2, ν_3 = 0,3,

IPE 200, A_1 = 2725 mm^2, $I_1 = I_{yy1}$ = 0,142·10^7 mm^4, k_{s1} = 0,388499,

IPE 140, A_2 = 1601 mm^2, $I_2 = I_{yy2}$ = 448461 mm^4, k_{s2} = 0,386579,

IPE 80, A_3 = 743 mm^2, $I_3 = I_{yy3}$ = 84676 mm^4, k_{s3} = 0,380084.

The results are $F_{\mathrm{buc\ sys\ analytical}}$ = 47024 N and $F_{\mathrm{buc\ sys\ numerical}}$ = 48983 N (Fig. 15). The deviation is +4,165 %.

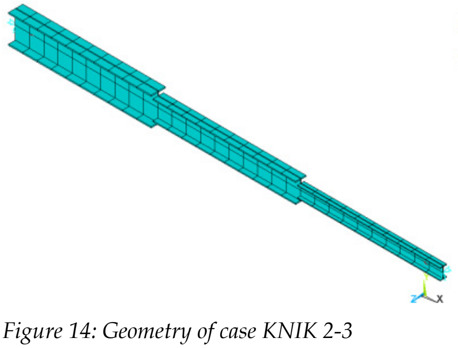


Figure 14: Geometry of case KNIK 2-3

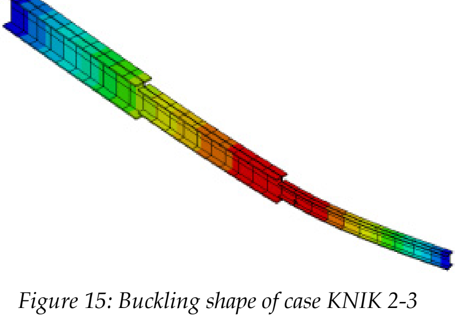


Figure 15: Buckling shape of case KNIK 2-3

Case Study KNIK 3-3

Case KNIK 3-3 considers a composite steel-wood beam of three segments (Fig. 16). The properties are L = 3000 mm, L_1 = 1000 mm, L_2 = 1000 mm, L_3 = 1000 mm,

$k_{\text{buc}} = \sqrt{3}$, $k_{\text{bc}} = 1$, $F_X = \pm 1$ N,

$E_1 = 2{,}1 \cdot 10^5$ N/mm², G_1 = 80769 N/mm², $\nu_1 = 0{,}3$,

$E_2 = 2{,}1 \cdot 10^5$ N/mm², G_2 = 80769 N/mm², $\nu_2 = 0{,}3$,

E_3 = 4500 N/mm², G_3 = 1731 N/mm², $\nu_3 = 0{,}3$,

$b \times h$ = 60 mm x 160 mm, A_1 = 9600 mm², $I_1 = I_{yy1} = 0{,}288 \cdot 10^7$ mm⁴, $k_{s1} = 0{,}842105$,

$b \times h$ = 30 mm x 80 mm, A_2 = 2400 mm², $I_2 = I_{yy2} = 0{,}18 \cdot 10^6$ mm⁴, $k_{s2} = 0{,}842105$,

$b \times h$ = 15 mm x 40 mm, A_3 = 600 mm², $I_3 = I_{yy3}$ = 11250 mm⁴, $k_{s3} = 0{,}842105$.

The results are $F_{\text{buc sys analytical}}$ = 166 N and $F_{\text{buc sys numerical}}$ = 171 N (Fig. 17). The deviation is +3,012 %.

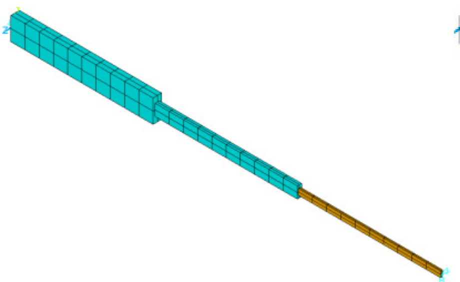


Figure 16: Geometry of case KNIK 3-3

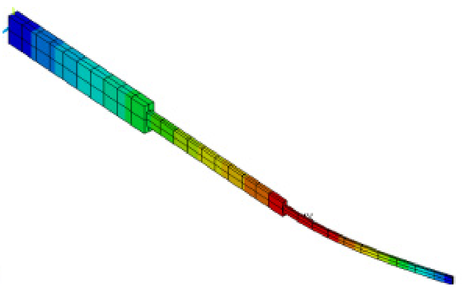


Figure 17: Buckling shape of case KNIK 3-3

5 Linear elastic lateral torsional buckling (kip)

5.1 *Model geometry*

The governing model geometry is shown in Figure 18. The following mechanical boundary (displacement and load) conditions are implemented (prescribed).

Boundary 1. $u_X = 0$, $u_Y = 0$, $u_Z = 0$, $\varphi_X = 0$,

$M_Z = -1$ (applied at centroid).

Boundary 2. $u_Y = 0$, $u_Z = 0$, $\varphi_X = 0$,

$M_Z = +1$ (applied at centroid).

The same notes apply as in Section 3.1.

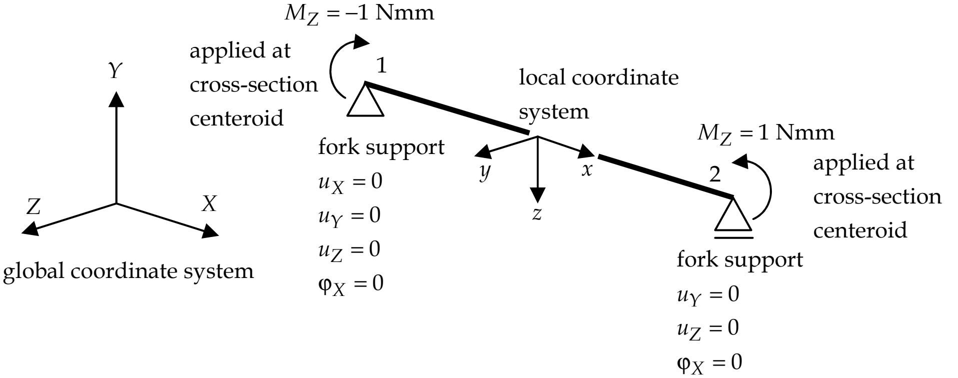


Figure 18: Coordinate systems and boundary conditions of the lateral torsional buckling model

5.2 *Analytical Model*

The model for lateral torsional buckling is mostly identical to the models for lateral buckling (Section 3.2). Not used in lateral torsional buckling is the shear deformation coefficient k_{si} . Extra is the is the torsional constant I_{ti} . The cross-sectional warping stiffness of segment i is assumed to be negligible. The engineering estimate of the buckling length $L_{\text{buc } i}$ of segment i is

$$L_{\text{buc } i} = k_{\text{buc}} L_i \tag{8}$$

where

k_{buc} is the buckling length factor, which is the same for each segment in the specific composite beam configuration.

$$k_{\text{buc}} = \frac{k_{bc} L}{\sum_{i=1}^{n} L_i} = \frac{k_{bc} \sum_{i=1}^{n} L_i}{\sum_{i=1}^{n} L_i} = k_{bc} \tag{9}$$

where

k_{bc} is the buckling length factor of the composite beam, dependent on the mechanical boundary conditions (support). In this specific case of a composite beam with span L and supported by two forks at the edges: $k_{bc} = 1$.

It is underlined that the buckling length of the composite beam is only dependent on the boundary conditions and hence not on cross-sectional properties. Substitution of Equation 9 in Equation 8 produces the following identity.

$$L_{\text{buc } i} = k_{\text{bc}} L_i = L_i \tag{10}$$

Taking into account the previous settled propositions, the linear elastic lateral torsional buckling moment $M_{\text{buc } i}$ of segment i is

$$\frac{1}{M_{\text{buc } i}} = \frac{1}{\dfrac{\pi}{k_{\text{buc}} L_i} \sqrt[2]{E_i I_i G_i I_{ti}}} = \frac{1}{\dfrac{\pi}{L_i} \sqrt[2]{E_i I_i G_i I_{ti}}} \tag{11}$$

The Föppl-Papkovich theorem (Tarnai 1995) is apt for this case. The following lateral torsional buckling system moment equation is written in universal layout.

$$\frac{1}{M_{\text{buc sys}}} = \sum_{i=1}^{n} \frac{1}{M_{\text{buc } i}} \tag{12}$$

where,

$M_{\text{buc sys}}$ is the linear elastic lateral torsional buckling moment of the composite beam (also symbolized as system).

n is the total number of segments.

The aforesaid procedure details the full linear elastic lateral torsional buckling analytical model.

6 Numerical verification (case studies) kip

Three case studies (KIP 1, KIP 2 and KIP 3) are consecutively undertaken and compendiously offered in order to inspect the legality or reliability of the analytical approach by judgment of numerical results.

Case study KIP 1-1

Case KIP 1-1 considers a wooden beam of one segment (Fig. 19).

The properties are L = 3000 mm, L_1 = 3000 mm, k_{buc} = 1, k_{bc} = 1, M_Z = ±1 Nmm,

E_1 = 4500 N/mm², G_1 = 1731 N/mm², ν_1 = 0,3,

b x h = 60 mm x 160 mm, A_1 = 9600 mm², $I_1 = I_{yy1}$ = 0,288·10⁷ mm⁴, I_{t1} = 0,90204·10⁷ mm⁴.

The results are $M_{\text{buc sys analytical}} = 0{,}149 \cdot 10^8$ Nmm and $M_{\text{buc sys numerical}} = 0{,}149 \cdot 10^8$ Nmm

(Fig. 20). The deviation is computed with $\frac{M_{\text{buc sys numerical}}}{M_{\text{buc sys analytical}}} 100\% - 100\% = 0\%$.

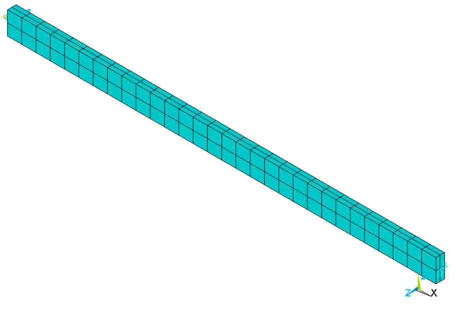


Figure 19: *Geometry of case KIP 1-1*

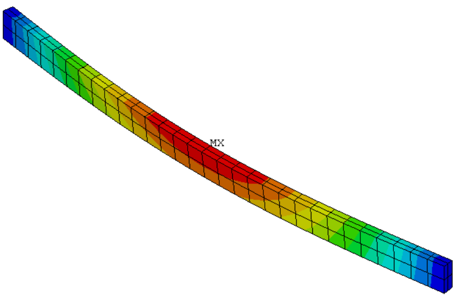


Figure 20: Buckling shape of case KIP 1-1

Case study KIP 1-2

Case KIP 1-2 considers a composite wooden beam of two segments (Fig. 21).

The properties are L = 3000 mm, L_1 = 1500 mm, L_2 = 1500 mm, $k_{\text{buc}} = 1$,

$k_{\text{bc}} = 1$, $M_Z = \pm 1$ Nmm,

E_1 = 4500 N/mm², G_1 = 1731 N/mm², $\nu_1 = 0{,}3$,

E_2 = 4500 N/mm², G_2 = 1731 N/mm², $\nu_2 = 0{,}3$,

b x h = 60 mm x 160 mm, A_1 = 9600 mm², $I_1 = I_{yy1}$ = 0,288·10⁷ mm⁴, I_{t1} = 0,90204·10⁷ mm⁴,

b x h = 30 mm x 80 mm, A_2 = 2400 mm², $I_2 = I_{yy2}$ = 0,18·10⁶ mm⁴, I_{t2} = 0,56377·10⁶ mm⁴.

The results are $M_{\text{buc sys analytical}} = 0{,}175 \cdot 10^7$ Nmm and $M_{\text{buc sys numerical}} = 0{,}176 \cdot 10^7$ Nmm

(Fig. 22). The deviation is +0,57%.

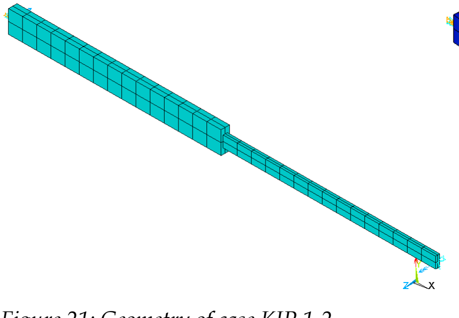


Figure 21: Geometry of case KIP 1-2

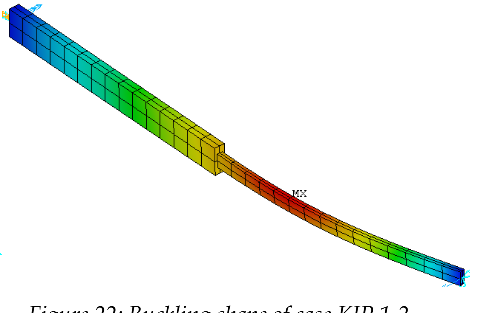


Figure 22: Buckling shape of case KIP 1-2

Case study KIP 1-3

Case KIP 2-3 considers a composite wooden beam of three segments (Fig. 23).

The properties are L = 3000 mm, L_1 = 1000 mm, L_2 = 1000 mm, L_3 = 1000 mm, k_{buc} = 1,

k_{bc} = 1, M_Z = ±1 Nmm,

E_1 = 4500 N/mm², G_1 = 1731 N/mm², ν_1 = 0,3,

E_2 = 4500 N/mm², G_2 = 1731 N/mm², ν_2 = 0,3,

E_3 = 4500 N/mm², G_3 = 1731 N/mm², ν_3 = 0,3,

b x h = 60 mm x 160 mm, A_1 = 9600 mm², $I_1 = I_{yy1}$ = $0{,}288 \cdot 10^7$ mm⁴, I_{t1} = $0{,}90204 \cdot 10^7$ mm⁴,

b x h = 30 mm x 80 mm, A_2 = 2400 mm², $I_2 = I_{yy2}$ = $0{,}18 \cdot 10^6$ mm⁴, I_{t1} = $0{,}56377 \cdot 10^6$ mm⁴,

b x h = 15 mm x 40 mm, A_3 = 600 mm², $I_3 = I_{yy3}$ = 11250 mm⁴, I_{t1} = 35236 mm⁴.

The results are $M_{\mathrm{buc\ sys\ analytical}}$ = 163701 Nmm and $M_{\mathrm{buc\ sys\ numerical}}$ = 164801 Nmm (Fig. 24). The deviation is +0,67 %.

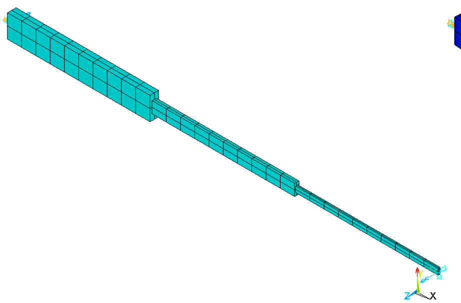


Figure 23: Geometry of case KIP 1-3

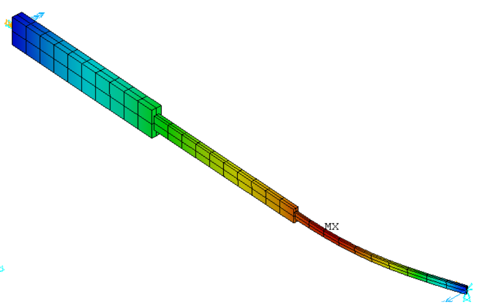


Figure 24: Buckling shape of case KIP 1-3

Case study KIP 2-1

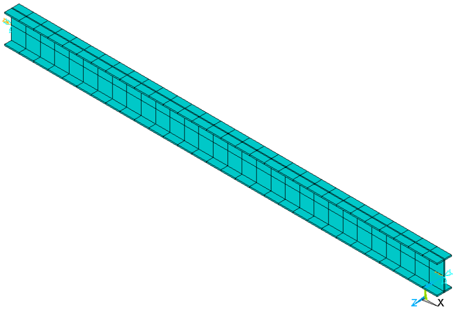
Case KIP 2-1 considers a steel beam of one segment (Fig. 25).

The properties are L = 3000 mm, L_1 = 3000 mm, k_{buc} = 1, k_{bc} = 1, M_Z = ±1 Nmm,

E_1 = 2,1·105 N/mm², G_1 = 80769 N/mm², ν_1 = 0,3.

IPE 200, A_1 = 2725 mm², $I_1 = I_{yy1}$ = $0{,}142 \cdot 10^7$ mm⁴, I_1 = 53176 mm⁴.

The obtained results are $M_{\mathrm{buc\ sys\ analytical}}$ = $37{,}48 \cdot 10^6$ Nmm and $M_{\mathrm{buc\ sys\ numerical}}$ = $37{,}5 \cdot 10^6$ Nmm (Fig. 25). The deviation is +0,053 %.



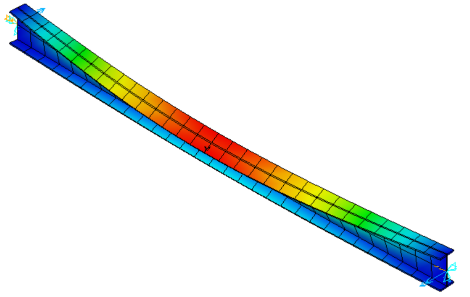


Figure 25: Geometry of case KIP 2-1 *Figure 26: Buckling shape of case KIP 2-1*

Case study KIP 2-2

Case KIP 2-2 considers a composite steel beam of two segments (Fig. 27). The properties are L = 3000 mm, L_1 = 1500 mm, L_2 = 1500 mm, k_{buc} = 1, k_{bc} = 1, M_Z = ±1 Nmm,

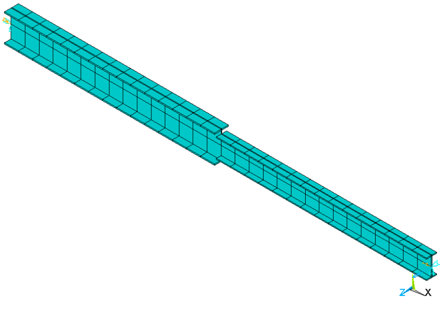
E_1 = 2,1·10^5 N/mm^2, G_1 = 80769 N/mm^2, ν_1 = 0,3,

E_2 = 2,1·10^5 N/mm^2, G_2 = 80769 N/mm^2, ν_2 = 0,3,

IPE 200, A_1 = 2725 mm^2, $I_1 = I_{yy1}$ = 0,142·10^7 mm^4, I_{t1} = 53176 mm^4,

IPE 140, A_2 = 1601 mm^2, $I_2 = I_{yy2}$ = 448461 mm^4, I_{t2} = 20978 mm^4.

The results are $M_{\text{buc sys analytical}}$ = 19,55·10^6 Nmm and $M_{\text{buc sys numerical}}$ = 19,2·10^6 Nmm (Fig. 28). The deviation is -1,79 %.



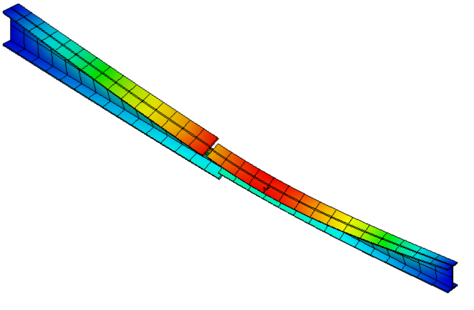


Figure 27: Geometry of case KIP 2-2 *Figure 28: Buckling shape of case KIP 2-2*

Case study KIP 2-3

Case KIP 2-3 considers a composite steel beam of three segments (Fig. 29).

The properties are L = 3000 mm, L_1 = 1000 mm, L_2 = 1000 mm, L_3 = 1000 mm,

k_{buc} = 1, k_{bc} = 1, M_Z = ±1 Nmm,

$E_1 = 2{,}1{\cdot}10^5$ N/mm², $G_1 = 80769$ N/mm², $\nu_1 = 0{,}3$,

$E_2 = 2{,}1{\cdot}10^5$ N/mm², $G_2 = 80769$ N/mm², $\nu_2 = 0{,}3$,

$E_3 = 2{,}1{\cdot}10^5$ N/mm², $G_3 = 80769$ N/mm², $\nu_3 = 0{,}3$,

IPE 200, $A_1 = 2725$ mm², $I_1 = I_{yy1} = 0{,}142{\cdot}10^7$ mm⁴, $I_{t1} = 53176$ mm⁴,

IPE 140, $A_2 = 1601$ mm², $I_2 = I_{yy2} = 448461$ mm⁴, $I_{t2} = 20978$ mm⁴,

IPE 80, $A_3 = 743$ mm², $I_3 = I_{yy3} = 84676$ mm⁴, $I_{t3} = 5773$ mm⁴.

The results are $M_{\text{buc sys analytical}} = 6{,}91{\cdot}10^6$ Nmm are $M_{\text{buc sys numerical}} = 6{,}68{\cdot}10^6$ Nmm (Fig. 30). The deviation is -3,328 %.

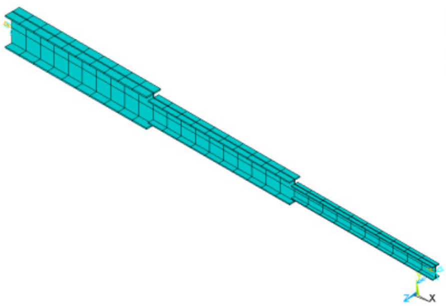


Figure 29: Geometry of case KIP 2-3

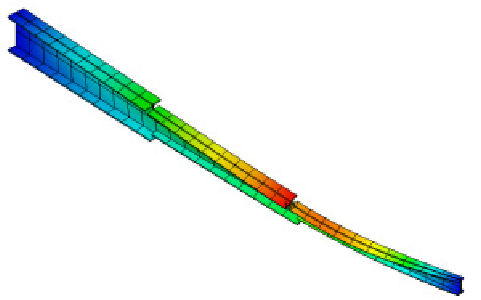


Figure 30: Buckling shape of case KIP 2-3

Case study KIP 3-3

Case KIP 3-3 considers a composite steel-wood beam of three segments (Fig. 31).

The properties are $L = 3000$ mm, $L_1 = 1000$ mm, $L_2 = 1000$ mm, $L_3 = 1000$ mm, $k_{\text{buc}} = 1$,

$k_{\text{bc}} = 1$, $M_Z = \pm 1$ Nmm,

$E_1 = 2{,}1{\cdot}10^5$ N/mm², $G_1 = 80769$ N/mm², $\nu_1 = 0{,}3$,

$E_2 = 2{,}1{\cdot}10^5$ N/mm², $G_2 = 80769$ N/mm², $\nu_2 = 0{,}3$,

$E_3 = 4500$ N/mm2, $G_3 = 1731$ N/mm², $\nu_3 = 0{,}3$,

b x h = 60 mm x 160 mm, $A_1 = 9600$ mm², $I_1 = I_{yy1} = 0{,}288{\cdot}10^7$ mm⁴, $I_{t1} = 0{,}90204{\cdot}10^7$ mm⁴,

b x h = 30 mm x 80 mm, $A_2 = 2400$ mm², $I_2 = I_{yy2} = 0{,}18{\cdot}10^6$ mm⁴, $I_{t2} = 0{,}56377{\cdot}10^6$ mm⁴,

b x h = 15 mm x 40 mm, $A_3 = 600$ mm², $I_3 = I_{yy3} = 11250$ mm⁴, $I_{t3} = 35236$ mm⁴,

The results are $M_{\text{buc sys analytical}} = 174324$ Nmm and $M_{\text{buc sys numerical}} = 175756$ Nmm (Fig. 32). The deviation is +0,821 %.

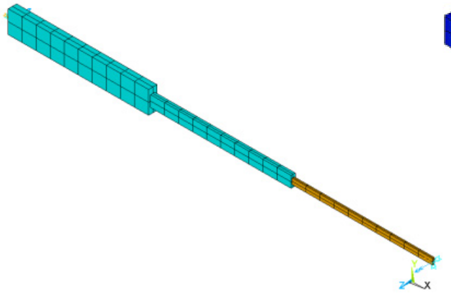


Figure 31: Geometry of case KIP 3-3

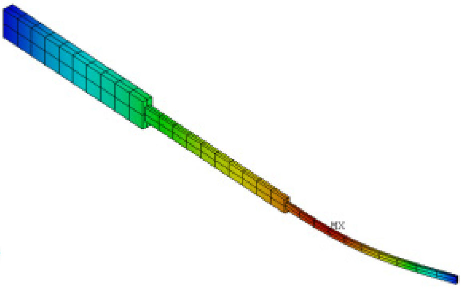


Figure 32: Buckling shape of case KIP 3-3

7 Conclusions

Numerical evaluation confirms the correctness of the presented linear elastic lateral shear and lateral torsional buckling analytical models. The deviations between the analytical and numerical approach are rather small and reveal clearly the ability to yield realistic outcomes.

The supplied analytical linear elastic buckling models produce accurate and reliable values, which can be classified as suitable for engineering design purposes.

Needless to cite that the analytical approach should always be cautiously applied and the results ought to be methodically checked by the responsible structural engineer.

A salient observation is that the presented tactic could also be invoked for linear elastic buckling assessment of castellated and cellular (steel) beams. However this assertion has not yet been verified and is thus speculative.

Insertion of plasticity is straightforward however treatment of this topic is beyond the scope of this article.

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